ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Dr. Helmut Bölcskei
Chair for Mathematical Information Science

Examination on Mathematics of Information
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Please note:

- Exam duration: 180 minutes
- Maximum number of points: 100
- You are allowed to use a printed annotated version of the lecture notes and of the exercise notes. Other documents as well as electronic devices (laptops, calculators, cellphones, etc...) are not allowed.
- Your solutions should be explained in detail and your handwriting needs to be clean and readable.
- Please do not use red or green pens. You may use pencils.
- Please note that the ETHZ "Disziplinarordnung RSETHZ 361.1" applies.

Before you start:

1. The problem statements consist of 7 pages including this page. Please verify that you have received all 7 pages.
2. Please fill in your name, student ID card number and sign below.
3. Please place your student ID card at the front of your desk so we can verify your identity.

During the exam:

4. For your solutions, please use only the empty sheets provided by us. Should you need additional sheets, please let us know.
5. Each problem consists of several subproblems. If you do not provide a solution to a subproblem, you may, whenever applicable, nonetheless assume its conclusion in the ensuing subproblems.

After the exam:

6. Please write your name on every sheet and prepare all sheets in a pile. All sheets, including those containing problem statements, must be handed in.
7. Please clean up your desk and remain seated and silent until you are allowed to leave the room in a staggered manner row by row.
8. Please avoid crowding and leave the building by the most direct route.

Family name: .... First name: ....

Legi-No.: ....

Number of additional sheets handed in: ....

Signature: ....

Problem 1 (25 points)

(a) Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Determine for each of the following sets whether it is a frame for $\mathcal{H}$ or not. For sets that are a frame, determine the tightest possible frame bounds A, B, else prove that the set is not a frame.

(i) (2 points)

$$\{h_k\}_{k\in\mathbb{N}} = \{(-1)^k e_k\}_{k\in\mathbb{N}} = \{-e_1, e_2, -e_3, e_4, \dots\}$$

(ii) (4 points)

$$\{h_k\}_{k\in\mathbb{N}} = \left\{e_1, \frac{1}{2}\,e_2, \frac{1}{2}\,e_2, \frac{1}{3}\,e_3, \frac{1}{3}\,e_3, \frac{1}{3}\,e_3, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \dots\right\}$$

(b) Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Define the set

$$\{g_k\}_{k\in\mathbb{N}} = \{e_k + e_{k+1}\}_{k\in\mathbb{N}} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, \dots\}.$$

(i) (4 points) Show that the set $\{g_k\}_{k\in\mathbb{N}}$ is complete for $\mathcal{H}$.
Hint: Recall that $\|x\| < \infty$, for all $x \in \mathcal{H}$.

(ii) (7 points) Show that the set $\{g_k\}_{k\in\mathbb{N}}$ is not a frame for $\mathcal{H}$.
Hint: It may be helpful to consider signals of the form $x_q = \sum_{\ell=1}^{\infty}(-q)^{\ell-1}e_\ell$ with $q \in (0,1)$.

(c) The Weyl operator $\mathbb{W}_{m,n}^{(T,F)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as

$$\mathbb{W}_{m,n}^{(T,F)} : f(\bullet) \to e^{2\pi i n F \bullet} f(\bullet - mT),$$

where $m, n \in \mathbb{N}$, and $T, F > 0$ are fixed time- and frequency-shift parameters, respectively.

(i) (4 points) Show that the adjoint operator of $\mathbb{W}_{m,n}^{(T,F)}$ is given by

$$\left(\mathbb{W}_{m,n}^{(T,F)}\right)^* = e^{-2\pi i n m T F}\, \mathbb{W}_{-m,-n}^{(T,F)}.$$

(ii) (4 points) Show that $\mathbb{W}_{m,n}^{(T,F)}$ is unitary by establishing that

$$\mathbb{W}_{m,n}^{(T,F)} (\mathbb{W}_{m,n}^{(T,F)})^* = (\mathbb{W}_{m,n}^{(T,F)})^*\, \mathbb{W}_{m,n}^{(T,F)} = \mathbb{I},$$

where $\mathbb{I}$ denotes the identity operator on $L^2(\mathbb{R})$.

Problem 2 (25 points)

Notation: For a vector $u \in \mathbb{C}^N$, a matrix $B \in \mathbb{C}^{m \times N}$, and a set $S \subset \{1, \ldots, N\}$, we define $u_S \in \mathbb{C}^{|S|}$ to be the vector obtained from u by keeping only the entries indexed by S, and similarly, we define $B_S \in \mathbb{C}^{m \times |S|}$ to be the matrix obtained from B by keeping only the columns indexed by S. Further, $S^c := \{1, \ldots, N\} \backslash S$ denotes the complement of the set S in $\{1, \ldots, N\}$. B^H stands for the conjugate transpose of the matrix B and $\mathcal{N}(B)$ refers to the null space of B (i.e., $\mathcal{N}(B) = \{v \in \mathbb{C}^N \mid Bv = 0\}$).

In compressed sensing, we are given a measurement vector $y \in \mathbb{C}^m$ obtained according to $y = Dx$, where $x \in \mathbb{C}^N$, $x \neq 0$, is the unknown (sparse) vector to be recovered and $D \in \mathbb{C}^{m \times N}$ is the so-called measurement matrix. In class, we studied two algorithms for recovering x from the observation y, namely

$$\arg\min_{\widehat{x}} \|\widehat{x}\|_0 \quad \text{subject to } D\widehat{x} = y \tag{P0}$$

and

$$\arg\min_{\widehat{x}} \|\widehat{x}\|_1 \quad \text{subject to } D\widehat{x} = y. \tag{P1}$$

(a) For this subproblem, we fix

$$D := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -\frac{4}{5} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad x := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and hence

$$y = Dx = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(i) (2 points) Compute $\mathcal{N}(D)$.

(ii) (1 point) Is the condition $\|x\|_0 < \frac{\operatorname{spark}(D)}{2}$ satisfied? Here, $\operatorname{spark}(D)$ is as in Definition 1 in the Handout.

(iii) (2 points) Is the condition

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right),$$

satisfied? Here, $\mu(D)$ denotes the coherence of D as in Definition 2 in the Handout.

(iv) (1 point) Specify the solution set for the linear system of equations $y = D\widehat{x}$, i.e., determine

$$\mathcal{X} := \{\widehat{x} \mid y = D\widehat{x}\}.$$

(v) (2 points) Is x uniquely recovered through (P0)?

(vi) (2 points) Is x uniquely recovered through (P1)?

(b) In the following subproblem, we establish sufficient conditions for recovery through (P1). Specifically, these conditions are in terms of the sign pattern of the vector $x \in \mathbb{C}^N$ to be recovered. We define the support set of x as $S = \{i \mid x_i \neq 0\} \subset \{1, \ldots, N\}$, and let the complex sign-function $\operatorname{sgn}(\bullet) : \mathbb{C}^N \to \mathbb{C}^N$ be given by

$$(\operatorname{sgn}(x))_k = \begin{cases} x_k/|x_k|, & \text{if } x_k \neq 0 \\ 0, & \text{else} \end{cases}.$$

Throughout we assume that $x \neq 0$.

(i) (7 points + 1 point for establishing the Hint) Show that x can be recovered through (P1) if it satisfies the following sufficient condition (C1):

$$\left| \sum_{j \in S} v_j \overline{(\operatorname{sgn}(x))_j} \right| < \sum_{k \in S^c} |v_k|, \qquad \text{for all } v \in \mathcal{N}(D) \setminus \{0\}. \tag{C1}$$

Hint: First show that $\forall u, v \in \mathbb{C}^N : |\langle u, v \rangle| \leq \|u\|_1 \|v\|_\infty$. (1 point)

(ii) (7 points) Show that x can be recovered through (P1) if it satisfies the following sufficient condition (C2):

$$\begin{gathered} \mathcal{N}(D_S) = \{0\} \qquad \text{and there exists } h \in \mathbb{C}^m \text{ s.t.} \\ (D^H h)_j = \operatorname{sgn}(x)_j,\ \forall j \in S, \qquad |(D^H h)_k| < 1,\ \forall k \in S^c. \end{gathered} \tag{C2}$$

Hint: Show that (C2) implies (C1).

Problem 3 (25 points)

In this problem, we derive a continuous-time version of an uncertainty relation presented in the lecture. Specifically, we consider a complex-valued signal $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of unit L^2-norm, i.e., $\|f\|_2 = 1$ and write $\hat{f}$ for its Fourier transform. We further introduce the time-limiting operator $P_{\mathcal{T}}$ and the frequency-limiting operator $P_{\mathcal{W}}$, defined as

$$(P_{\mathcal{T}} f)(t) = \mathbb{1}_{\mathcal{T}}(t) f(t) \quad \text{and} \quad (P_{\mathcal{W}} f)(t) = \int_{\mathcal{W}} e^{2\pi i w t} \hat{f}(w) dw,$$

where $\mathcal{T}$ and $\mathcal{W}$ are bounded subsets of $\mathbb{R}$, and $\mathbb{1}_{\mathcal{T}}$ is the indicator of $\mathcal{T}$, i.e.,

$$\mathbb{1}_{\mathcal{T}}(t) = \begin{cases} 1, & \text{if } t \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Further, the signal f considered is $\varepsilon_{\mathcal{T}}$-concentrated to $\mathcal{T}$ and $\varepsilon_{\mathcal{W}}$-concentrated to $\mathcal{W}$ according to

$$\|f - P_{\mathcal{T}} f\|_2 \leq \varepsilon_{\mathcal{T}} \quad \text{and} \quad \|f - P_{\mathcal{W}} f\|_2 \leq \varepsilon_{\mathcal{W}}.$$

For the operator P, we write $\|P\|_{2\to 2} := \sup_{\|g\|_2 = 1} \|Pg\|_2$ for its operator norm.

(a) (6 points) Show that

$$\|f - P_{\mathcal{W}} P_{\mathcal{T}} f\|_2 \leq \varepsilon_{\mathcal{T}} + \varepsilon_{\mathcal{W}}.$$

Hint: First prove and then use that $\|P_{\mathcal{W}}\|_{2\to 2} = 1$.

(b) (3 points) Show that

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2} \geq 1 - \varepsilon_{\mathcal{T}} - \varepsilon_{\mathcal{W}}.$$

Hint: You can use, without proof, the reverse triangle inequality, namely that, for all $g, h \in L^2(\mathbb{R})$, one has $\|g - h\|_2 \geq \|g\|_2 - \|h\|_2$.

(c) (8 points) Show that, for all $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|g\|_2 = 1$, we have

$$(P_{\mathcal{W}} P_{\mathcal{T}} g)(s) = \int_{-\infty}^{\infty} q(s, t) g(t) dt,$$

for some $q(s, t)$ to be expressed explicitly, and use this relation to prove that

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2}^2 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s, t)|^2 dt\, ds.$$

Hint: You can use, without proof, Fubini's theorem (cf. Handout).

(d) (6 points) Prove the following identity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q(s,t)|^2 dt\, ds = |\mathcal{W}||\mathcal{T}|.$$

<u>Hint</u>: *First express the function q in terms of the inverse Fourier transform of an indicator function and then use the Plancherel identity.*

(e) (2 points) Combine the results established in the previous subproblems to prove that

$$|\mathcal{W}||\mathcal{T}| \geq \left(1 - (\varepsilon_{\mathcal{T}} + \varepsilon_{\mathcal{W}})\right)^2.$$

Problem 4 (25 points)

Given a compact set $K \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, we define the Minkowski dimension of K with respect to the norm $\|\cdot\|$ as

$$\dim_{\|\cdot\|}(K) := \lim_{\varepsilon\to 0^+} \frac{\log_2 \mathcal{N}(\varepsilon; K, \|\cdot\|)}{\log_2(1/\varepsilon)}, \tag{1}$$

where $\mathcal{N}(\varepsilon; K, \|\cdot\|)$ denotes the ε-covering number of K with respect to the norm $\|\cdot\|$, and $\varepsilon \in (0, 1)$. We will only consider compact sets K for which the limit (1) exists.

(a) (i) (3 points) Fix $x \in \mathbb{R}^n$ and show that

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the usual 1- and 2-norm, respectively.

(ii) (4 points) Show that the result in (a)(i) implies the following inequalities between the corresponding ε-covering numbers

$$\mathcal{N}(\varepsilon; K, \|\cdot\|_2) \le \mathcal{N}(\varepsilon; K, \|\cdot\|_1) \le \mathcal{N}(\varepsilon/\sqrt{n}; K, \|\cdot\|_2). \tag{2}$$

(iii) (3 points) Deduce from (2) that

$$\dim_{\|\cdot\|_1}(K) = \dim_{\|\cdot\|_2}(K).$$

(iv) (5 points) Show that the Minkowski dimension of K is independent of the choice of the norm on $\mathbb{R}^n$, i.e., given two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathbb{R}^n$, we have

$$\dim_{\|\cdot\|}(K) = \dim_{\|\cdot\|'}(K).$$

We will denote this common quantity by $\dim(K)$, without subscript, hereafter and refer to it simply as "the Minkowski dimension".
Hint: Use the equivalence of norms in finite dimensions (cf. Handout).

(b) (i) (5 points) Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, prove that the Minkowski dimension of the ball $B_{\|\cdot\|}(0, R)$ (with respect to the norm $\|\cdot\|$) centered at the origin and of radius $R > 0$ satisfies $\dim(B_{\|\cdot\|}(0, R)) = n$, where the unsubscripted quantity $\dim(\cdot)$ is as defined in subproblem (a)(iv).
Hint: First prove the result in the case $R = 1$ using the relation between metric entropy and the volume ratio (cf. Handout). Then argue, for general $R > 0$, that $\dim(B_{\|\cdot\|}(0, R)) = \dim(B_{\|\cdot\|}(0, 1))$, using, without proof, the scaling relation $\mathcal{N}(\varepsilon; B_{\|\cdot\|}(0, 1), \|\cdot\|) = \mathcal{N}(R\varepsilon; B_{\|\cdot\|}(0, R), \|\cdot\|)$, for all $\varepsilon > 0$.

(ii) (3 points) Show that the Minkowski dimension of every compact set $K \subset \mathbb{R}^n$ is bounded according to $\dim(K) \le n$.
Hint: Use the result from subproblem (b)(i).

(iii) (2 points) Provide an example of a compact set $K \subset \mathbb{R}^n$ with $\dim(K) < n$.