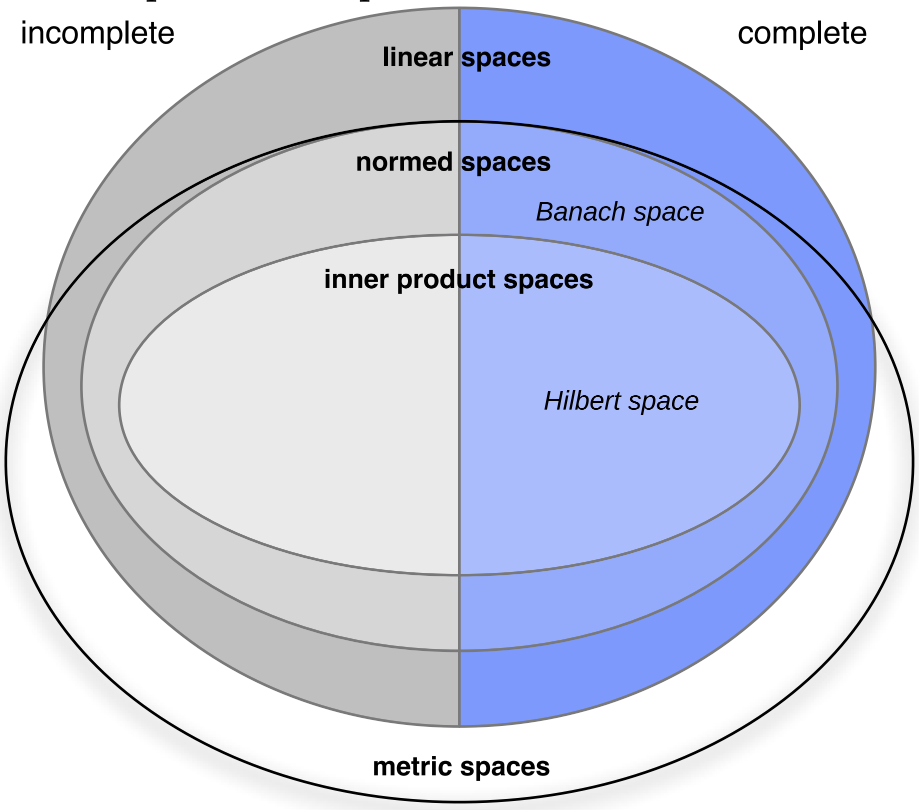
Vector Space Concepts



Scalar Field. For all linear (vector) spaces in the following, the scalar field will be either the field of real numbers, $\mathcal{F} = \mathbb{R}$, or the complex field, $\mathcal{F} = \mathbb{C}$.

Normed Space [1, 2, §2]. A *norm* $\|\cdot\|$ on a linear space $(\mathcal{U}, \mathcal{F})$ is a mapping $\|\cdot\| : \mathcal{U} \to [0, \infty)$ that satisfies, for all $\mathbf{u}, \mathbf{v} \in \mathcal{U}, \alpha \in \mathcal{F}$,

1. $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{0}$.
2. $\|\alpha\mathbf{u}\| = |\alpha| \, \|\mathbf{u}\|$.
3. *Triangle inequality:* $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

A norm defines a metric $d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|$ on $\mathcal{U}$. A *normed (linear) space* $(\mathcal{U}, \|\cdot\|)$ is a linear space $\mathcal{U}$ with a norm $\|\cdot\|$ defined on it.

- The norm is a continuous mapping of $\mathcal{U}$ into $\mathbb{R}_+$.
- A norm $\|\cdot\|$ on a linear space $\mathcal{U}$ is said to be *equivalent* to a norm $\|\cdot\|_0$ on $\mathcal{U}$ if there are positive numbers a and b such that $a\|\mathbf{u}\|_0 \le \|\mathbf{u}\| \le b\|\mathbf{u}\|_0$ for all $\mathbf{u} \in \mathcal{U}$. Equivalent norms define the same topology on $\mathcal{U}$.
- The metric d induced by a norm is *translation invariant*, i.e., it satisfies
  - $d(\mathbf{u} + \mathbf{x}, \mathbf{v} + \mathbf{x}) = d(\mathbf{u}, \mathbf{v})$,
  - $d(\alpha\mathbf{u} + \alpha\mathbf{v}) = |\alpha| \, d(\mathbf{u}, \mathbf{v})$

  for all $\mathbf{u}, \mathbf{v}, \mathbf{x} \in \mathcal{U}$ and $\alpha \in \mathcal{F}$.
- *Riesz's Lemma:* Let $\mathcal{Y}$ and $\mathcal{Z}$ be linear subspaces of a normed space $\mathcal{U}$ and let $\mathcal{Y}$ be a closed proper subset of $\mathcal{U}$. Then, for every $\theta \in (0, 1)$, there is a $\mathbf{z} \in \mathcal{Z}$ such that $\|\mathbf{z} - \mathbf{y}\| \ge \theta$ for $\|\mathbf{z}\| = 1$ and for all $\mathbf{y} \in \mathcal{Y}$.
- A subset $\mathcal{T}$ of a normed space $\mathcal{U}$ is said to be *total* in $\mathcal{U}$ if span $\mathcal{T}$ is dense in $\mathcal{U}$.
- Let $\mathcal{S}$ be a linear subspace of a normed space $\mathcal{U}$. If $\mathcal{S}$ is open as a subset in $\mathcal{U}$, then $\mathcal{S} = \mathcal{U}$.

Basis and Dimension [2, 1].

- Let $\mathcal{V}$ be a linear space. A linearly independent subset $\mathcal{S} \subset \mathcal{V}$ that spans $\mathcal{V}$ is called a *Hamel basis* for $\mathcal{V}$.
  - Every linear space has a Hamel basis, so that every nonzero $\mathbf{v} \in \mathcal{V}$ has a unique representation as a linear combination of finitely many elements of $\mathcal{S}$ with nonzero scalar coefficients.
  - If $\mathcal{B}_1$ and $\mathcal{B}_2$ are Hamel basis for a linear space $\mathcal{V}$, then they have the same cardinality.
- The *dimension* $\dim \mathcal{V}$ of a linear space $\mathcal{V}$ is defined as the cardinality of any Hamel basis of $\mathcal{V}$.
  - If $\dim \mathcal{V}$ is finite, $\mathcal{V}$ is called a *finite-dimensional* linear space.
  - A linear space $\mathcal{V}$ is *finite dimensional* iff there is a positive integer N such that $\mathcal{V}$ contains a linearly independent set of N vectors whereas any set of $N+1$ vectors of $\mathcal{V}$ is linearly dependent.
  - If $\mathcal{V}_1$ and $\mathcal{V}_2$ are linear spaces over the same scalar field, then they are isomorphic iff $\dim \mathcal{V}_1 = \dim \mathcal{V}_2$.
- If a normed space $\mathcal{U}$ contains a sequence $\{\mathbf{e}_n\}$ with the property that for every $\mathbf{u} \in \mathcal{U}$ there is a unique sequence of scalars $\{\alpha_n\}$ such that $\|\mathbf{u} - (\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \dots + \alpha_N\mathbf{e}_N\| \to 0$ as $N \to \infty$, then $\{\mathbf{e}_n\}$ is called a *Schauder basis* for $\mathcal{U}$. A Schauder basis is different from a Hamel basis in that a countably infinite number of basis vectors and scalar coefficients may be needed to uniquely represent a given vector.

Convergence [1, §4.8]. Let $\{\mathbf{u}_n\}$ be a sequence of vectors in a normed space $\mathcal{U}$.

- The sequence $\{\mathbf{u}_n\}$ is said to be *strongly convergent*, or *convergent in norm*, if there is a $\mathbf{u} \in \mathcal{U}$, called the *strong limit* of $\{\mathbf{u}_n\}$, such that $\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\| = 0$. Strong convergence is written $\mathbf{u}_n \to \mathbf{u}$ and often referred to simply as *convergence*.
- The sequence $\{\mathbf{u}_n\}$ is said to be *weakly convergent* if there is a $\mathbf{u} \in \mathcal{U}$, called the *weak limit* of $\{\mathbf{u}_n\}$, such that $\lim_{n\to\infty} f(\mathbf{u}_n) = f(\mathbf{u})$ for every bounded linear functional f on $\mathcal{U}$, i.e., for every f in the dual space $\mathcal{U}'$. Weak convergence is written $\mathbf{u}_n \xrightarrow{w} \mathbf{u}$.
  - The weak limit $\mathbf{u}$ is unique.
  - Every subsequence of $\{\mathbf{u}_n\}$ converges weakly to $\mathbf{u}$.
  - The sequence $\{\|\mathbf{u}_n\|\}$ is bounded.
- Strong convergence implies weak convergence to the same limit.
- If $\dim \mathcal{U} < \infty$, then weak convergence implies strong convergence.
- The *(infinite) series* $\mathbf{u}_1 + \mathbf{u}_2 + \dots$ is said to converge (strongly) if the sequence of *partial sums* $\mathbf{s}_n := \mathbf{u}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_n$ converges, i.e., if $\mathbf{s}_n \to \mathbf{s}$ for some $\mathbf{s} \in \mathcal{U}$.
- The above series is said to be *absolutely convergent* if the infinite series $\|\mathbf{u}_1\| + \|\mathbf{u}_2\| + \dots$ converges.
- A series is said to be *unconditionally convergent* if (i) it is convergent for each possible rearrangement of terms, and (ii) if each rearrangement converges to the same limit.

Banach Space [1, §2]. A *Banach space* $(\mathcal{B}, \|\cdot\|)$ is a complete normed space, complete in the metric induced by its norm $\|\cdot\|$.

- A linear subspace $\mathcal{S}$ of a Banach space $\mathcal{B}$ is a Banach space, i.e., it is complete, iff $\mathcal{S}$ is closed in $\mathcal{B}$.
- For a series on a Banach space, absolute convergence implies strong convergence and unconditional convergence.
- Let $(\mathcal{U}, \|\cdot\|)$ be a normed space. Then there is a Banach space $\mathcal{B}$ and an isometry f from $\mathcal{B}$ onto a linear subspace $\mathcal{S} \subset \mathcal{B}$ that is dense in $\mathcal{B}$. The space $\mathcal{B}$ is unique except for isometries. Thus, every normed space can be *completed*.

Finite-Dimensional Normed Spaces.

- Every finite-dimensional linear subspace $\mathcal{S}$ of a normed space $\mathcal{U}$ is complete; in particular, every finite-dimensional normed space is complete.
- Every finite-dimensional linear subspace of a normed space $\mathcal{U}$ is closed in $\mathcal{U}$ and separable.
- On a finite-dimensional linear space, all norms are equivalent.
- In a finite-dimensional normed space $\mathcal{U}$, any subset $\mathcal{S} \subset \mathcal{U}$ is compact iff $\mathcal{S}$ is closed and bounded.

Inner Product Space [1, §3]. Let $(\mathcal{G}, \mathcal{F})$ be a linear space. An *inner product* is a mapping $\langle\cdot,\cdot\rangle : \mathcal{G} \times \mathcal{G} \to \mathcal{F}$ that satisfies the following properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and scalars $\alpha \in \mathcal{F}$:

1. $\langle\mathbf{x} + \mathbf{y}, \mathbf{z}\rangle = \langle\mathbf{x}, \mathbf{z}\rangle + \langle\mathbf{y}, \mathbf{z}\rangle$.
2. $\langle\alpha\mathbf{x}, \mathbf{y}\rangle = \alpha\langle\mathbf{x}, \mathbf{y}\rangle$.
3. $\langle\mathbf{x}, \mathbf{y}\rangle = \langle\mathbf{y}, \mathbf{x}\rangle^*$.
4. $\langle\mathbf{x}, \mathbf{x}\rangle \ge 0$ with equality iff $\mathbf{x} = \mathbf{0}$.

A linear space $\mathcal{G}$ on which an inner product $\langle\cdot,\cdot\rangle$ is defined is called an *inner product space* $(\mathcal{G}, \langle\cdot,\cdot\rangle)$.

- An inner product defines a norm $\|\mathbf{x}\| := \sqrt{\langle\mathbf{x}, \mathbf{x}\rangle}$ and a metric $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{y} - \mathbf{x}\| = \sqrt{\langle\mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x}\rangle}$ on $\mathcal{G}$. Hence, inner product spaces are normed spaces.
- The inner product is called *sesquilinear*, because it is linear in the first term and conjugate linear in the second term: $\langle\mathbf{x}, \alpha\mathbf{y}\rangle = \alpha^*\langle\mathbf{x}, \mathbf{y}\rangle$.
- The inner product satisfies the *Schwarz inequality:* $|\langle\mathbf{x}, \mathbf{y}\rangle| \le \|\mathbf{x}\|\|\mathbf{y}\|$.
- The induced norm satisfies the *triangle inequality:* $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ with equality iff $\mathbf{y} = c\mathbf{x}$ for some positive scalar c.
- The induced norm satisfies the *parallelogram equality:* $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$.
- Continuity: if in an inner product space $\mathcal{G}$ $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$, then $\langle\mathbf{x}_n, \mathbf{y}_n\rangle \to \langle\mathbf{x}, \mathbf{y}\rangle$, where $\{\mathbf{x}_n\}, \mathbf{x}, \{\mathbf{y}_n\}, \mathbf{y} \in \mathcal{G}$.
- If $\langle\mathbf{x}_1, \mathbf{y}\rangle = \langle\mathbf{x}_2, \mathbf{y}\rangle$ for all $\mathbf{y}$ in an inner product space, then $\mathbf{x}_1 = \mathbf{x}_2$.

Two inner product spaces $\mathcal{G}$ and $\mathcal{V}$ are called *unitarily equivalent* if there is an isomorphism $\mathbb{U} : \mathcal{G} \to \mathcal{V}$ of $\mathcal{G}$ onto $\mathcal{V}$ that preserves inner products, i.e., $\langle\mathbb{U}\mathbf{u}_1, \mathbb{U}\mathbf{u}_2\rangle = \langle\mathbf{u}_1, \mathbf{u}_2\rangle$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{G}$. The mapping $\mathbb{U}$ is called a *unitary operator*.

Orthogonality [2, 1]. An element $\mathbf{x}$ of an inner product space $\mathcal{G}$ is said to be *orthogonal* to an element $\mathbf{y} \in \mathcal{G}$, denoted $\mathbf{x} \perp \mathbf{y}$, if $\langle\mathbf{x}, \mathbf{y}\rangle = 0$. Similarly, for $\mathcal{A}, \mathcal{B} \subset \mathcal{G}$, $\mathbf{x} \perp \mathcal{A}$ means that $\mathbf{x} \perp \mathbf{a}$ for all $\mathbf{a} \in \mathcal{A}$, and $\mathcal{A} \perp \mathcal{B}$ means that $\mathbf{a} \perp \mathbf{b}$ for all $\mathbf{a} \in \mathcal{A}$ and all $\mathbf{b} \in \mathcal{B}$.

- An *orthogonal set* $\mathcal{O}$ in an inner product space $\mathcal{G}$ is a subset $\mathcal{O} \subset \mathcal{G}$ whose elements are pairwise orthogonal. An *orthonormal set* is an orthogonal set whose elements have unit norm. A countable orthogonal (orthonormal) set is called an *orthogonal* (orthonormal) sequence.
- An orthogonal set is linearly independent.
- Let $\{\mathbf{e}_\alpha\}$ be an orthonormal set in an inner product space $\mathcal{G}$, and let $\mathbf{g}$ be any point in $\mathcal{G}$. Then $\langle\mathbf{g}, \mathbf{e}_\alpha\rangle$ is nonzero for at most a countable number of vectors $\mathbf{e}_\alpha$.
- Let $\mathcal{G}$ be an inner product space and $\mathcal{C}$ a nonempty convex subset of $\mathcal{G}$ that is complete in the metric induced by the inner product. Then, for every $\mathbf{g} \in \mathcal{G}$ there exists a unique $\mathbf{c}_0 \in \mathcal{C}$ such that $\inf_{\mathbf{c}\in\mathcal{C}} \|\mathbf{g} - \mathbf{c}\| = \|\mathbf{g} - \mathbf{c}_0\|$. If $\mathcal{C}$ is a complete linear subspace of $\mathcal{G}$, then $(\mathbf{g} - \mathbf{c}_0) \perp \mathcal{C}$.
- *Bessel inequality:* Let $\{\mathbf{e}_n\}$ be an orthonormal sequence in an inner product space $\mathcal{G}$. Then, for every $\mathbf{g} \in \mathcal{G}$,
$$\sum_{n=1}^{\infty} |\langle\mathbf{g}, \mathbf{e}_n\rangle|^2 \le \|\mathbf{g}\|^2.$$

Orthogonal Complement [2, 1, 3]. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets in an inner product space $\mathcal{G}$. The set $\mathcal{A}^\perp := \{\mathbf{g} \in \mathcal{G} : \mathbf{g} \perp \mathcal{A}\}$ is called the *orthogonal complement* of $\mathcal{A}$ in $\mathcal{G}$.

- The orthogonal complement $\mathcal{A}^\perp$ of $\mathcal{A}$ in $\mathcal{G}$ is a closed linear subspace of $\mathcal{G}$. If $\mathcal{G}$ is complete, then $\mathcal{A}^\perp$ is complete.
- If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{B}^\perp \subset \mathcal{A}^\perp$.
- $\mathcal{A} \subset (\mathcal{A}^\perp)^\perp$.
- If $\mathbf{g} \in \mathcal{A} \cap \mathcal{A}^\perp$, then $\mathbf{g} = \mathbf{0}$.
- If $\mathcal{A} \subset \mathcal{G}$, then $\mathcal{A}^\perp = ((\mathcal{A}^\perp)^\perp)^\perp$.
- $\{\mathbf{0}\}^\perp = \mathcal{G}$ and $\mathcal{G}^\perp = \{\mathbf{0}\}$.
- If $\mathcal{A}$ is a dense subset of $\mathcal{G}$, then $\mathcal{A}^\perp = \{\mathbf{0}\}$.
- If $\{\mathcal{A}_n\}$ is a sequence of subspaces, then $\left(\overline{\operatorname{span}\{\mathcal{A}_n\}}\right)^\perp = \cap_n \mathcal{A}_n^\perp$, and $(\cap_n \mathcal{A}_n)^\perp = \overline{\operatorname{span}\{\mathcal{A}_n^\perp\}}$.

An orthonormal set $\mathcal{O}$ in an inner product space $\mathcal{G}$ that is total in $\mathcal{G}$ is called a *total orthonormal set*, or sometimes a *maximal* or *complete* orthonormal set.

- Let $\mathcal{O} \subset \mathcal{G}$ be a subset of an inner product space $\mathcal{G}$. Then, if $\mathcal{O}$ is total in $\mathcal{G}$, there does not exist a nonzero vector $\mathbf{g} \in \mathcal{G}$ that is orthogonal to every element of $\mathcal{O}$.
- If $\mathcal{G}$ is complete, i.e., a Hilbert space, the above condition is sufficient for $\mathcal{O}$ to be total in $\mathcal{G}$.

Hilbert Space [1, §3]. A complete inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space*. Thus, a Hilbert space is a Banach space on which an inner product is defined.

- For any inner product space $\mathcal{G}$ there exists a Hilbert space $\mathcal{H}$ and an isomorphism from $\mathcal{G}$ onto a dense linear subspace $\mathcal{D} \subset \mathcal{H}$. The space $\mathcal{H}$ is uniquely except for isomorphisms. Thus, every inner product space can be *completed*.
- Let $\{\mathbf{h}_n\}$ be a sequence in a Hilbert space $\mathcal{H}$. Then, $\mathbf{h}_n \xrightarrow{w} \mathbf{h}$ iff $\langle\mathbf{h}_n, \mathbf{z}\rangle \to \langle\mathbf{h}, \mathbf{z}\rangle$ for all $\mathbf{z} \in \mathcal{H}$.
- In every Hilbert space $\mathcal{H} \ne \{\mathbf{0}\}$, there exists a total orthonormal set.
- An orthonormal set $\mathcal{O}$ in a Hilbert space $\mathcal{H}$ is total in $\mathcal{H}$ iff for all $\mathbf{h} \in \mathcal{H}$ the *Parseval relation* holds:
$$\sum_{\mathbf{e}\in\mathcal{O}} |\langle\mathbf{h}, \mathbf{e}\rangle|^2 = \|\mathbf{h}\|^2.$$
- A total orthonormal sequence, i.e., a countable total orthonormal set, in a Hilbert space $\mathcal{H}$ is called an *orthonormal basis* for $\mathcal{H}$.
- If a Hilbert space $\mathcal{H}$ is separable, every total orthonormal set is countable, i.e., every total orthonormal set is an orthonormal basis. Conversely, if $\mathcal{H}$ contains an orthonormal sequence that is total in $\mathcal{H}$, then $\mathcal{H}$ is separable. Thus, there exists an orthonormal basis for $\mathcal{H}$ iff $\mathcal{H}$ is separable.
- All total orthonormal sets in a given Hilbert space have the same cardinality, called the *Hilbert dimension* of $\mathcal{H}$.
- Two Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, both over the same scalar field, are isomorphic iff they have the same Hilbert dimension.
- Let $\mathcal{Y}$ be any closed linear subspace of a Hilbert space $\mathcal{H}$. Then, $\mathcal{H} = \mathcal{Y} \oplus \mathcal{Z}$, where $\mathcal{Z} = \mathcal{Y}^\perp$ is the orthogonal complement of $\mathcal{Y}$. Each $\mathbf{h} \in \mathcal{H}$ can be uniquely represented as $\mathbf{h} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in \mathcal{Y}$ and $\mathbf{z} \in \mathcal{Z} = \mathcal{Y}^\perp$, and $\|\mathbf{h}\| = \|\mathbf{y}\| + \|\mathbf{z}\|$.
- Let $\mathcal{S} \subset \mathcal{H}$ be a linear subspace of $\mathcal{H}$; then, $(\mathcal{S}^\perp)^\perp = \overline{\mathcal{S}}$. If $\mathcal{S}$ is closed, then $(\mathcal{S}^\perp)^\perp = \mathcal{S}$.
- For any nonempty susbspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, span $\mathcal{S}$ is dense in $\mathcal{H}$ iff $\mathcal{S}^\perp = \{\mathbf{0}\}$. If $\mathcal{S}$ is closed and $\mathcal{S}^\perp = \{\mathbf{0}\}$, then $\mathcal{S} = \mathcal{H}$.

Let $\mathcal{Y}$ and $\mathcal{Z}$ be two subspaces of $\mathcal{H}$. The *canonical correlation* $\rho(\mathcal{Y}, \mathcal{Z})$ between these two subspaces is defined as
$$\rho(\mathcal{Y}, \mathcal{Z}) := \sup\{|\langle\mathbf{y}, \mathbf{z}\rangle| : \mathbf{y} \in \mathcal{Y}, \mathbf{z} \in \mathcal{Z}, \|\mathbf{y}\| = \|\mathbf{z}\| = 1\}$$
and the *angle* $\theta(\mathcal{Y}, \mathcal{Z})$ between these subspaces as $\theta(\mathcal{Y}, \mathcal{Z}) = \cos\rho(\mathcal{Y}, \mathcal{Z})$.

- Let $\mathbf{y} \in \mathcal{Y}$, $\mathbf{z} \in \mathcal{Z}$. Then, the following conditions are equivalent:
  - $\rho(\mathcal{Y}, \mathcal{Z}) < 1$, i.e., $\theta(\mathcal{Y}, \mathcal{Z}) > 0$.
  - $\inf\{\|\mathbf{y} - \mathbf{z}\| : \|\mathbf{y}\| = \|\mathbf{z}\| = 1\} > 0$.
  - There is a constant c such that $\|\mathbf{y}\| \le c\|\mathbf{y} + \mathbf{z}\|$ for all $\mathbf{y}$, $\mathbf{z}$.
  - The direct sum $\mathcal{Y} \oplus \mathcal{Z}$ is a closed subspace of $\mathcal{H}$.

Fourier Series [2]. *Riesz-Fischer Theorem:* Let $\{\mathbf{e}_n\}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$, and let $\{\alpha_n\}$ be a sequence of scalars. Then, the series

$$\mathbf{h} = \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$$

converges in norm iff $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. In this case, the coefficients α_n are called the *Fourier coefficients* of $\mathbf{h}$, and they are given as $\alpha_n = \langle \mathbf{h}, \mathbf{e}_n \rangle$. Conversely, the above series always converges to $\mathbf{h}$ if the α_n are the Fourier coefficients of any $\mathbf{h} \in \mathcal{H}$.

- The above series is convergent iff it converges unconditionally.

Let $\{\mathbf{e}_n\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$, then the following statements are equivalent:

- The set $\{\mathbf{e}_n\}$ is an orthonormal basis for $\mathcal{H}$.
- For any $\mathbf{h} \in \mathcal{H}$, the *Fourier series expansion* of $\mathbf{h}$ is given as $\mathbf{h} = \sum_n \alpha_n \mathbf{e}_n$, where $\alpha_n = \langle \mathbf{h}, \mathbf{e}_n \rangle$.
- *Parseval equality:* For any $\mathbf{x}, \mathbf{y} \in \mathcal{H}$,
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_n \langle \mathbf{x}, \mathbf{e}_n \rangle \langle \mathbf{y}, \mathbf{e}_n \rangle^*.$$
- For any $\mathbf{h} \in \mathcal{H}$,
$$\|\mathbf{h}\|^2 = \sum_n |\langle \mathbf{h}, \mathbf{e}_n \rangle|^2.$$
- Let $\mathcal{M}$ be any linear subspace of $\mathcal{H}$ that contains $\{\mathbf{e}_n\}$; then $\mathcal{M}$ is dense in $\mathcal{H}$.

Banach Algebra [4, 5]. Strictly speaking, a *Banach Algebra* is an algebra $\mathcal{B}$ over a scalar field $\mathcal{F}$, where $\mathcal{B}$ is also a Banach space under a norm $\|\cdot\|$ that satisfies the multiplicative inequality $\|\mathbf{x}\mathbf{y}\| \le \|\mathbf{x}\|\|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}$.
In the following, though, an associative unit *complex Banach algebra*, i.e., a Banach algebra over the complex field $\mathbb{C}$ that is associative and contains an identity element $\mathbf{1}$ with respect to vector multiplication such that $\|\mathbf{1}\| = 1$ is simply referred to as a complex Banach algebra.

- An element $\mathbf{b} \in \mathcal{B}$ is called *invertible* if $\mathbf{b}$ has an inverse in $\mathcal{B}$. The invertible elements of $\mathcal{B}$ form a group with respect to vector multiplication.
- Let $\mathcal{S} \subset \mathcal{B}$ denote the set of all invertible elements of $\mathcal{B}$. If $\mathbf{b} \in \mathcal{B}$ and $\|\mathbf{b}\| < 1$, then,
  - $\mathbf{1} + \mathbf{b} \in \mathcal{S}$,
  - $(\mathbf{1} + \mathbf{b})^{-1} = \sum_{n=0}^{\infty} (-1)^n \mathbf{b}^n$,
  - $\|(\mathbf{1} + \mathbf{b})^{-1} - \mathbf{1} + \mathbf{b}\| \le \|\mathbf{b}\|^2/(1 - \|\mathbf{b}\|)$.
  - The set $\mathcal{S}$ is open, and the mapping $\mathbf{b} \to \mathbf{b}^{-1}$ is a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$.
- The *spectrum* $\mathcal{S}(\mathbf{b})$ of an element $\mathbf{b} \in \mathcal{B}$ is defined as the set of all complex numbers λ such that $\mathbf{b} - \lambda\mathbf{1}$ is not invertible.
- Let f be a bounded linear functional on $\mathcal{B}$. Then, for any fixed $\mathbf{b} \in \mathcal{B}$, the function $g(\lambda) := f((\mathbf{b} - \lambda\mathbf{1})^{-1})$, $\lambda \notin \mathcal{S}(\mathbf{b})$, is holomorphic in the complement of $\mathcal{S}(\mathbf{b})$, and $g(\lambda) \to 0$ as $\lambda \to \infty$.
- For every $\mathbf{b} \in \mathcal{B}$, the spectrum $\mathcal{S}(\mathbf{b})$ is compact and not empty.
- If each nonzero element of $\mathcal{B}$ is invertible, then the complex Banach algebra $\mathcal{B}$ is isometrically isomorphic to the complex field $\mathbb{C}$. This also implies that $\mathcal{B}$ is commutative.
- For any $\mathbf{b} \in \mathcal{B}$, the *spectral radius* $r_{\mathbf{b}}$ of $\mathbf{b}$ is defined as $r_{\mathbf{b}} := \sup\{|\lambda| : \lambda \in \mathcal{S}(\mathbf{b})\}$; it can be computed as $r_= \lim_{n\to\infty} \|\mathbf{b}^n\|^{1/n}$.

A complex-valued *homomorphism* f on a Banach algebra $\mathcal{B}$ is a linear functional that preserve vector multiplication, i.e., a functional f for which $f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$ and $f(\mathbf{x}\mathbf{y}) = f(\mathbf{x}) f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ and $\alpha, \beta \in \mathcal{F}$. Furthermore, f is not identical to 0. Let $\mathcal{M}$ denote the set of all complex-valued homomorphisms f of $\mathcal{B}$. Then,

- $\lambda \in \mathcal{S}(\mathbf{b})$ iff $f(\mathbf{b}) = \lambda$ for some $f \in \mathcal{M}$.
- The vector $\mathbf{b}$ is invertible in $\mathcal{B}$ iff $f(\mathbf{b}) \neq 0$ for every $f \in \mathcal{M}$.
- $f(\mathbf{b}) \in \mathcal{S}(\mathbf{b})$ for every $\mathbf{b} \in \mathcal{B}$ and $f \in \mathcal{M}$.
- $|f(\mathbf{b})| \le r_{\mathbf{b}} \le \|\mathbf{b}\|$ for every $\mathbf{b} \in \mathcal{B}$ and $f \in \mathcal{M}$.

Some Important Linear Spaces

Euclidean Space [6]. The N-dimensional complex *Euclidean space*

$$\mathbb{C}^N := \left\{ \mathbf{x} : \mathbf{x} = [x_0\, x_1 \cdots x_{N-1}]^T, x_n \in \mathbb{C} \right\}$$

with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{n=0}^{N-1} x_n y_n^*$$

and corresponding induced norm is a finite-dimensional Hilbert space.

Sequence Space [3]. The *sequence space*

$$l^p := \left\{ \mathbf{x} : \mathbf{x} = \{x_n\}_{n=0}^{\infty}, \quad x_n \in \mathbb{C}, \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}$$

with norm

$$\|\mathbf{x}\|_p := \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}$$

is a Banach space for $1 \le p \le \infty$.

- For $p = \infty$, the norm is the supremum norm: $\|\mathbf{x}\|_\infty := \sup_n |x_n|$.
- An important subspace of l^∞ is the space whose elements are sequences that decay to zero, i.e., $x_n \to 0$ as $n \to \infty$.
- For $p = 2$, the space l^2 with inner product
$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{n=0}^{\infty} x_n y_n^*$$
and norm $\|\mathbf{x}\|_2 := \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is an infinite-dimensional Hilbert space, called the *Hilbert sequence space*.

Space of Continuous Functions. Let $\mathcal{C}[a, b]$ denote the space of all complex-valued continuous functions $f : [a, b] \to \mathbb{C}$ with pointwise addition and scalar multiplication.

- $\mathcal{C}^\infty[a, b]$, endowed with the supremum norm $\|f\|_\infty := \sup_{a \le t \le b} |f(t)|$, is a Banach space.
- Endowed with the inner product $\langle f, g \rangle := \int_a^b f(t) g^*(t) dt$ and the induced norm, this space is an inner product space but not a Hilbert space.

Lebesgue Space [3, 7]. Let $\mathcal{X}$ be an arbitrary set, $\mathscr{F}$ the σ-algebra of subsets of $\mathcal{X}$, and μ a nonnegative measure on $\mathscr{F}$. The *Lebesgue space*

$$\mathcal{L}^p(\mathcal{X}, \mathscr{F}, \mu) := \left\{ f : \mathcal{X} \to \mathbb{C} \quad \text{measurable}, \int |f|^p d\mu < \infty \right\}$$

with norm

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{1/p}$$

is a Banach space for $1 \le p < \infty$.

- For $p = \infty$, the space $\mathcal{L}^\infty(\mathcal{X}, \mathscr{F}, \mu)$ with μ-essential supremum norm $\|f\|_\infty := \text{ess}_\mu \sup_t |f(t)|$ is also a Banach space.
- If μ is finite and $\mathcal{X} = (a, b]$, the spaces $\mathcal{L}^p(\mu) := \mathcal{L}^p((a, b], \mathscr{F}, \mu)$ are nested: $\mathcal{L}^p(\mu) \subseteq \mathcal{L}^q(\mu)$ for $p \ge q$.
- The space $\mathcal{L}^2(\mathcal{X}, \mathscr{F}\mu)$ with inner product $\langle f, g \rangle := \int_{\mathcal{X}} f g^* d\mu$ and induced norm is a Hilbert space, called the *Hilbert function space*. The elements of $\mathcal{L}^2(\mathcal{X}, \mathscr{F}, \mu)$ are equivalence classes of functions that differ on null sets.
- The space $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ is a Hilbert space. It is a dense subspace of $\mathcal{L}^2(\mathbb{R})$.
- For $\mathcal{X} = \{0, 1, \ldots, N-1\}$ or $\mathcal{X} = \mathbb{Z}_+$ and μ the counting measure on the collection $\mathscr{F}$ of all subsets of $\mathcal{X}$, the space $\mathcal{L}^2(\mathcal{X}, \mathscr{F}\mu)$ reduces to $\mathbb{C}^N$ or l^2, respectively.
- When μ is a probability measure, i.e., $\mu(\mathcal{X}) = 1$ for arbitrary $\mathcal{X}$, then $\mathcal{L}^2(\mathcal{X}, \mathscr{F}, \mu)$ is the space of all random variables with finite second moment.

Schwarz Space [7]. The *Schwarz space* $\mathcal{S}$ is the space of all infinitely differentiable, rapidly decaying functions of a real parameter t:

$$\mathcal{S} := \left\{ f : \mathbb{R} \to \mathbb{C} : \lim_{t\to\infty} t^m \frac{d^n f(t)}{dt^n} = 0 \quad \forall m, n \in \mathbb{N} \right\}$$

Paley-Wiener Space. to write

Sobolev Space. to write

Hardy Space [3]. Let $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open disk in the complex plane. For $0 < p < \infty$, the space

$$\mathcal{H}^p := \left\{ f : f \text{ analytic in } \mathcal{D}, \quad \sup_{0 \le r < 1} \int \left| f(re^{i\lambda}) \right|^p d\lambda < \infty \right\}$$

with norm

$$\|f\|_p := \sup_{0 \le r < 1} \left(\int \left| f(re^{i\lambda}) \right|^p d\lambda \right)^{1/p}$$

is a Banach space, called the *Hardy space*.

Reproducing Kernel Hilbert Spaces. to write

Linear Operators and Linear Functionals

Linear Operator [1, 2]. A linear operator $\mathbb{T}$ is a mapping of a linear space $\mathcal{V}$ into a linear space $\mathcal{Z}$ such that

1. The domain $\mathcal{D}(\mathbb{T})$ is a linear space $\mathcal{V}$, and the range $\mathcal{R}(\mathbb{T})$ lies in a linear space $\mathcal{Z}$ over the same scalar field $\mathcal{F}$.
2. For all $\mathbf{v}, \mathbf{u} \in \mathcal{V}$ and scalars α,
$$\mathbb{T}(\mathbf{v} + \mathbf{u}) = \mathbb{T}\mathbf{v} + \mathbb{T}\mathbf{u},$$
$$\mathbb{T}(\alpha\mathbf{v}) = \alpha\mathbb{T}\mathbf{v}.$$

The *null space* $\mathcal{N}(\mathbb{T})$ of $\mathbb{T}$ is the set of all $\mathbf{v} \in \mathcal{D}(\mathbb{T})$ such that $\mathbb{T}\mathbf{v} = \mathbf{0}$. The null space is a linear space.

- The range space $\mathcal{R}(\mathbb{T})$ of a linear operator is a linear space.
- Two linear operators $\mathbb{T}$ and $\mathbb{S}$ are said to be *equal* if they have the same domain and if $\mathbb{T}\mathbf{v} = \mathbb{S}\mathbf{v}$ for all $\mathbf{v} \in \mathcal{D}(\mathbb{T}) = \mathcal{D}(\mathbb{S})$.
- $\dim \mathcal{D}(\mathbb{T}) = N < \infty \implies \dim \mathcal{R}(\mathbb{T}) \le N$.
- The dimensions of the null space $\mathcal{N}(\mathbb{T})$,, the range space $\mathcal{R}(\mathbb{T})$ and the space $\mathcal{X}$ itself are related as $\dim \mathcal{N}(\mathbb{T}) + \dim \mathcal{R}(\mathbb{T}) = \dim \mathcal{X}$.
- Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ be linear spaces over the same scalar field so that $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isomorphic. The linear operators $\mathbb{T}_1 : \mathcal{X}_1 \to \mathcal{Y}_1$ and $\mathbb{T}_2 : \mathcal{X}_2 \to \mathcal{Y}_2$ are said to be *isomorphically equivalent* if there exists isomorphisms $\mathbb{U} : \mathcal{X}_1 \to \mathcal{X}_2$ and $\mathbb{W} : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $\mathbb{T}_1 = \mathbb{W}^{-1}\mathbb{T}_2\mathbb{U}$ and $\mathbb{T}_2 = \mathbb{W}\mathbb{T}_1\mathbb{U}^{-1}$.
- Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be isomorphic linear spaces. The linear operators $\mathbb{T}_1 : \mathcal{X}_1 \to \mathcal{X}_1$ and $\mathbb{T}_2 : \mathcal{X}_2 \to \mathcal{X}_2$ are said to be *similar* if there exists an isomorphism $\mathbb{U} : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\mathbb{T}_1 = \mathbb{U}^{-1}\mathbb{T}_2\mathbb{U}$ and $\mathbb{T}_2 = \mathbb{U}\mathbb{T}_1\mathbb{U}^{-1}$.
- Let $\mathbb{T} : \mathcal{V} \to \mathcal{V}$ be a linear operator and $\mathcal{M} \subset \mathcal{V}$ a linear subspace of $\mathcal{V}$ such that $\mathbb{T}(\mathcal{M}) \subset \mathcal{M}$; then $\mathcal{M}$ is called *invariant under* $\mathbb{T}$. In this case, the restriction of $\mathbb{T}$ to $\mathcal{M}$ is a mapping of $\mathcal{M}$ into itself.
- Let $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$. If some closed linear subspace $\mathcal{M} \subset \mathcal{H}$ and its orthogonal complement $\mathcal{M}^\perp$ are invariant under $\mathbb{T}$, then $\mathcal{M}$ is said to *reduce* $\mathbb{T}$.
- Any operator that maps a Banach space onto another Banach space is an *open mapping*.

Inverse Operator. Let $\mathbb{T} : \mathcal{V} \to \mathcal{Z}$ be a linear operator. Then, the *inverse operator* $\mathbb{T}^{-1} : \mathcal{R}(\mathbb{T}) \to \mathcal{D}(\mathbb{T})$ exists iff $\mathbb{T}\mathbf{v} = \mathbf{0}$ implies that $\mathbf{v} = \mathbf{0}$.

- If $\mathbb{T}^{-1}$ exists, it is a linear operator.
- If $\dim \mathcal{D}(\mathbb{T}) = N < \infty$ and $\mathbb{T}^{-1}$ exists, then $\dim \mathcal{R}(\mathbb{T}) = \dim \mathcal{D}(\mathbb{T})$.
- An invertible linear operator is a homeomorphism.
- Let $\mathbb{T} : \mathcal{X} \to \mathcal{Y}$ and $\mathbb{S} : \mathcal{Y} \to \mathcal{Z}$ be bijective linear operators, where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are linear spaces. Then, the inverse $(\mathbb{S}\mathbb{T})^{-1} : \mathcal{Z} \to \mathcal{X}$ of the composition (also called *product*) $\mathbb{S}\mathbb{T} := \mathbb{S} \circ \mathbb{T}$ exists and $(\mathbb{S}\mathbb{T})^{-1} = \mathbb{T}^{-1}\mathbb{S}^{-1}$.
- A bounded bijective operator $\mathbb{T} : \mathcal{X} \to \mathcal{Y}$ between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ has a bounded inverse.
- *Von Neumann Theorem:* Let $\mathbb{T} : \mathcal{B} \to \mathcal{B}$ be a bounded operator on a Banach space $\mathcal{B}$ that satisfies $\|\mathbb{I} - \mathbb{T}\| < 1$. Then, $\mathbb{T}$ is invertible, and $\mathbb{T}^{-1} = \sum_{n=0}^{\infty} (\mathbb{I} - \mathbb{T})^n$. Furthermore, $\|\mathbb{T}^{-1}\| \le 1/(1 - \|\mathbb{I} - \mathbb{T}\|)$.

Projections [2]. A linear operator $\mathbb{P} : \mathcal{X} \to \mathcal{X}$ that satisfies $\mathbb{P}^2 = \mathbb{P}$ is called a *projection*.

- Range $\mathcal{R}(\mathbb{P})$ and null space $\mathcal{N}(\mathbb{P})$ are disjoint subspaces of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{R}(\mathbb{P}) + \mathcal{N}(\mathbb{P}) = \mathcal{R}(\mathbb{P}) \oplus \mathcal{N}(\mathbb{P})$, i.e., $\mathcal{R}(\mathbb{P})$ and $\mathcal{N}(\mathbb{P})$ are algebraic complements of one another.
- If $\mathbb{P}$ is a projection, so is $\mathbb{I} - \mathbb{P}$, and $\mathcal{R}(\mathbb{P}) = \mathcal{N}(\mathbb{I} - \mathbb{P})$ and $\mathcal{N}(\mathbb{P}) = \mathcal{R}(\mathbb{I} - \mathbb{P})$.
- Let $\mathcal{S} \subset \mathcal{X}$ be a subspace of $\mathcal{X}$. Then there exists a projection $\mathbb{P} : \mathcal{X} \to \mathcal{X}$ such that $\mathcal{R}(\mathbb{P}) = \mathcal{S}$.
- Given two disjoint subspaces $\mathcal{V}$ and $\mathcal{U}$ with $\mathcal{X} = \mathcal{U} \oplus \mathcal{V}$, there is a unique projection $\mathbb{P}$ such that $\mathcal{R}(\mathbb{P}) = \mathcal{U}$ and $\mathcal{N}(\mathbb{P}) = \mathcal{V}$.

Finite-Dimensional Spaces [1, §2.9, §7.1]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional linear spaces over the same field $\mathcal{F}$, with $\dim \mathcal{X} = N, \dim \mathcal{Y} = K$. Let $\mathcal{E} := \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be a basis for $\mathcal{X}$, and let $\mathcal{B} := \{\mathbf{b}_1, \ldots, \mathbf{b}_K\}$ be a basis for $\mathcal{Y}$.

- Any linear operator $\mathbb{T} : \mathcal{X} \to \mathcal{Y}$ is uniquely determined by the K images of the N basis vectors $\mathbf{y}_k = \mathbb{T}\mathbf{e}_n$.
- Any linear operator $\mathbb{T}$ on a finite-dimensional linear space can be represented by a matrix $\mathbf{T}$ with $[\mathbf{T}]_{k,n} = t_{k,n}$, where $\mathbf{T}$ depends on the bases $\mathcal{E}$ and $\mathcal{B}$. Hence, the image of any vector $\mathbf{x} \in \mathcal{X}$ can be obtained as
$$\mathbf{y} = \mathbb{T}\mathbf{x} = \sum_{k=1}^{K} \sum_{n=1}^{N} (t_{k,n}\xi_n)\mathbf{b}_k$$
where $\mathbf{x} = \sum_{n=1}^{N} \xi_n \mathbf{e}_n$.
- For given bases $\mathcal{E}$ and $\mathcal{B}$, the matrix $\mathbf{T}$ is uniquely determined by $\mathbb{T}$.
- Conversely, any $K \times N$ matrix $\mathbf{T}$ defines a linear operator with respect to given bases for$\mathcal{X}$ and $\mathcal{Y}$.
- Two matrices that represent a linear operator on a finite-dimensional normed space relative to two different bases are similar.

Linear Functionals [1, 2]. A *linear functional* is a linear operator $f : \mathcal{V} \to \mathcal{F}$, defined on some linear space $\mathcal{V}$, whose range is in the scalar field $\mathcal{F}$ of the linear space.

- *Hahn-Banach Theorem:* Let $\mathcal{V}$ be a real or complex linear space, and let g be a real-valued functional on $\mathcal{V}$ that is subadditive, i.e., $g(\mathbf{u} + \mathbf{v}) \le g(\mathbf{u}) + g(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, and that satisfies $g(\alpha\mathbf{u}) = |\alpha| g(\mathbf{v})$ for every scalar α. Let f be a linear functional, defined on a subspace $\mathcal{Z}$ of $\mathcal{V}$, that satisfies $|f(\mathbf{z})| \le g(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{Z}$. Then, f has a linear extension $\tilde{f}$ from $\mathcal{Z}$ to $\mathcal{V}$ that satisfies $|f(\tilde{\mathbf{v}})| \le g(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$.
- The codimension of $\mathcal{N}(f)$ is 1.
- If $\mathcal{A}$ is any subspace of $\mathcal{V}$ with $\mathcal{N}(f) \subset \mathcal{A}$ and $\mathcal{N}(f) \neq 0$, then $\mathcal{A} = \mathcal{V}$.
- For some linear functional f and some scalar α, the set $\{\mathbf{v} \in \mathcal{V} : f(\mathbf{v}) = \alpha\}$ is called the *hyperplane* in $\mathcal{V}$ determined by f and α.

Algebraic Dual [1]. The set $\mathcal{V}^\star$ of all linear functionals defined on a linear space $\mathcal{V}$ is itself a linear space, called the *algebraic dual space* of $\mathcal{V}$. Its vector sum is defined as $s(\mathbf{v}) = (f_1 + f_2)(\mathbf{v}) := f_1(\mathbf{v}) + f_2(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$, and the product of a scalar α and a vector, i.e., a functional $f \in \mathcal{V}^\star$, is defined for all $\mathbf{v} \in \mathcal{V}^\star$ as $p(\mathbf{v}) = (\alpha f)(\mathbf{v}) := \alpha f(\mathbf{v})$.

- Let $\mathcal{V}$ be an N-dimensional linear space, and let $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be a basis for $\mathcal{V}$. Define the set of linear functionals $\mathcal{B} := \{f_1, \ldots, f_N\}$ with $f_k(\mathbf{e}_n) = \delta_{kn}$. Then $\mathcal{B}$ is a basis for the algebraic dual space $\mathcal{V}^\star$ of $\mathcal{V}$, and $\dim \mathcal{E} = \dim \mathcal{B}$; $\mathcal{B}$ is called the *dual basis* of $\mathcal{E}$.

Linear Functionals on Normed Spaces

Linear Functionals [1, 2]. Let $f : \mathcal{U} \to \mathcal{F}$ be a linear functional on a normed space $\mathcal{U}$.

- The *norm* $\|f\|$ of a linear functional f is the usual operator norm: $\|f\| = \sup_{\mathbf{u} \in \mathcal{U}, \mathbf{u} \neq 0} |f(\mathbf{u})|$.
- A *bounded linear functional* is a linear function f that satisfies $\|f\| \le a$ for some $a \in \mathbb{R}$.
- On a normed space $\mathcal{U}$, the Hahn-Banach Theorem implies that every bounded linear functional f on a subspace $\mathcal{S} \subset \mathcal{U}$ has a linear extension $\tilde{f}$ on $\mathcal{U}$ that has the same norm,
$$\sup_{\mathbf{u} \in \mathcal{U}, \|\mathbf{u}\| = 1} \left| \tilde{f}(\mathbf{u}) \right| = \sup_{\mathbf{s} \in \mathcal{S}, \|\mathbf{s}\| = 1} |f(\mathbf{s})|.$$
- Let $\mathcal{U}$ be a normed space and let $\mathbf{u} \in \mathcal{U}$. Then, there exists a bounded linear functional f on $\mathcal{U}$ such that $\|f\| = 1$ and $f(\mathbf{u}) = \|\mathbf{u}\|$.

Sesquilinear Form [1, §3.8]. Let $\mathcal{V}$ and $\mathcal{Z}$ be liner spaces over the same scalar field $\mathcal{F}$. A *sesquilinear form*, or *sesquilinear function* f on $\mathcal{V}\times\mathcal{Z}$ is a mapping $f:\mathcal{V}\times\mathcal{Z}\to\mathcal{F}$ such that for all $\mathbf{v},\mathbf{v}_1,\mathbf{v}_2\in\mathcal{V}$ and $\mathbf{z},\mathbf{z}_1,\mathbf{z}_2\in\mathcal{Z}$ and all scalars α and β

- $f(\mathbf{v}_1+\mathbf{v}_2,\mathbf{z}) = f(\mathbf{v}_1,\mathbf{z}) + f(\mathbf{v}_2,\mathbf{z})$,
- $f(\mathbf{v},\mathbf{z}_1+\mathbf{z}_2) = f(\mathbf{v},\mathbf{z}_1) + f(\mathbf{v},\mathbf{z}_2)$,
- $f(\alpha\mathbf{v},\mathbf{z}) = \alpha f(\mathbf{v},\mathbf{z})$,
- $f(\mathbf{v},\beta\mathbf{z}) = \beta^* f(\mathbf{v},\mathbf{z})$.

Dual Space [1]. Let $\mathcal{U}$ be a normed space. Then the set of all bounded linear functionals on $\mathcal{U}$ constitutes a normed space under the usual operator norm $\|f\| = \sup_{\mathbf{u}\in\mathcal{U},\|\mathbf{u}\|=1}|f(\mathbf{u})|$. This space is called the *dual space* $\mathcal{U}'$ of $\mathcal{U}$.

- The dual space $\mathcal{U}'$ of a normed space $\mathcal{U}$ is a Banach space, whether or not $\mathcal{U}$ is complete.
- For every $\mathbf{u}$ in a normed space $\mathcal{U}$,
$$\|\mathbf{u}\| = \sup_{\substack{f\in\mathcal{U}'\\ f\neq 0}} \frac{|f(\mathbf{u})|}{\|f\|}.$$
- Given a linearly independent set $\{f_1,\dots,f_N\}\in\mathcal{U}'$, there are elements $\mathbf{u}_1,\dots,\mathbf{u}_N$ in $\mathcal{U}$ such that $f_j(\mathbf{u}_k) = \delta_{jk}$.

Convergence [1, §4.9]. For linear functionals, strong and weak convergence are equivalent, so that a sequence $\{f_n\}$ of bounded linear functionals on a normed space $\mathcal{U}$ is said to be

- *strongly convergent* if there is an $f\in\mathcal{U}'$, called the *strong limit* of $\{f_n\}$, such that $\|f_n - f\|\to 0$; this is written as $f_n\to f$;
- *weak* convergent* if there is an $f\in\mathcal{U}'$, called the *weak* limit* of $\{f_n\}$, such that $f_n(\mathbf{u})\to f(\mathbf{u})$ for all $\mathbf{u}\in\mathcal{U}$; this is written as $f_n \xrightarrow{w^*} f$.

Linear Operators on Normed and Banach Spaces

Continuity [2, 1]. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let $\mathbb{T}:\mathcal{X}\to\mathcal{Y}$ be a linear operator.

- The operator $\mathbb{T}$ is continuous iff
$$\mathbb{T}\Big(\sum_{n=1}^{\infty}\alpha_n\mathbf{x}_n\Big) = \sum_{n=1}^{\infty}\alpha_n\mathbb{T}(\mathbf{x}_n)$$
for every convergent series $\sum_{n=1}^{\infty}\alpha_n\mathbf{x}_n$ in $\mathcal{X}$.
- If $\mathbb{T}$ is continuous at a single point, it is continuous.
- The linear operator $\mathbb{T}$ is continuous iff it is bounded.
- If a linear operator $\mathbb{T}$ is continuous, it is uniformly continuous.
- If $\mathcal{X}$ is finite dimensional, then $\mathbb{T}$ is continuous.

Operator Norm [1, §2.7]. Let $\mathbb{T}:\mathcal{U}\to\mathcal{Z}$ be a linear operator that maps a normed space $\mathcal{U}$ into a normed space $\mathcal{Z}$. The *operator norm* is defined as
$$\|\mathbb{T}\| := \sup_{\substack{\mathbf{u}\in\mathcal{U}\\ \mathbf{u}\neq\mathbf{0}}}\frac{\|\mathbb{T}\mathbf{u}\|}{\|\mathbf{u}\|}$$
where the norms on the RHS are vector norms in $\mathcal{Z}$ and $\mathcal{U}$. If $\mathcal{D}(\mathbb{T}) = \{\mathbf{0}\}$, then $\|\mathbb{T}\| := 0$.

- The operator norm $\|\mathbb{T}\|$ of $\mathbb{T}$ is equivalent to
$$\|\mathbb{T}\| = \sup_{\substack{\mathbf{u}\in\mathcal{U}\\ \|\mathbf{u}\|=1}}\|\mathbb{T}\mathbf{u}\|.$$

Bounded Linear Operators [1, §2.7]. The linear operator $\mathbb{T}:\mathcal{U}\to\mathcal{Z}$ that maps a normed space $\mathcal{U}$ into a normed space $\mathcal{Z}$ is said to be *bounded* if there is a real number a such that $\|\mathbb{T}\|\leq a$.

- A linear operator $\mathbb{T}$ is bounded iff it is continuous.
- If a normed space $\mathcal{U}$ is finite dimensional, then every linear operator on $\mathcal{U}$ is bounded.
- $\mathbb{T} = 0$ iff $\langle\mathbb{T}\mathbf{u},\mathbf{z}\rangle = 0$ for all $\mathbf{u}\in\mathcal{U}$ and $\mathbf{z}\in\mathcal{Z}$.
- The null space $\mathcal{N}(\mathbb{T})$ of $\mathbb{T}$ is closed.
- If $\{\mathbf{u}_n\}$ a sequence in $\mathcal{D}(\mathbb{T})$, then $\mathbf{u}_n\to\mathbf{u}$ implies $\mathbb{T}\mathbf{u}_n\to\mathbb{T}\mathbf{u}$.
- For bounded linear operators $\mathbb{T}_1:\mathcal{X}\to\mathcal{Y}$ and $\mathbb{T}_2:\mathcal{Y}\to\mathcal{Z}$ on normed spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, it follows that $\|\mathbb{T}_1\mathbb{T}_2\|\leq\|\mathbb{T}_1\|\|\mathbb{T}_2\|$, and for $\mathbb{T}:\mathcal{X}\to\mathcal{X}$ that $\|\mathbb{T}^n\|\leq\|\mathbb{T}\|^n$.
- *Uniform Boundedness Theorem:* Let $\{\mathbb{T}_n\}$ be a sequence of linear operators $\mathbb{T}_n:\mathcal{B}\to\mathcal{U}$ from a Banach space $\mathcal{B}$ into a normed space $\mathcal{U}$ such that $\|\mathbb{T}_n\mathbf{b}\|\leq c_{\mathbf{b}}<\infty$ for every $\mathbf{b}\in\mathcal{B}$ and every $n=1,2,\dots$. Then, the sequence of norms $\{\|\mathbb{T}_n\|\}$ is bounded, i.e., there exists a c such that $\|\mathbb{T}_n\|\leq c$ for all $n=1,2,\dots$.
- A bounded linear operator $\mathbb{T}$ from a Banach space $\mathcal{B}$ onto a Banach space $\mathcal{Z}$ has the property that the image $\mathbb{T}(\mathcal{B}_1(\mathbf{0}))$ of the open unit ball around the origin contains an open ball around $\mathbf{0}\in\mathcal{Z}$.
- *Open mapping theorem:* A bounded linear operator $\mathbb{T}$ from a Banach space onto another Banach space is an open mapping. Hence, if $\mathbb{T}$ is bijective, $\mathbb{T}^{-1}$ is continuous and thus bounded.

Operator Topologies [1, 2]. Let $\mathcal{G}(\mathcal{U},\mathcal{Z})$ denote the set of all bounded linear operators from a normed space $\mathcal{U}$ into a normed space $\mathcal{Z}$ over the same scalar field. The set $\mathcal{G}(\mathcal{U},\mathcal{Z})$ is a linear space under *operator addition* $(\mathbb{T}_1+\mathbb{T}_2)\mathbf{u} := \mathbb{T}_1\mathbf{u}+\mathbb{T}_2\mathbf{u}$, for all $\mathbf{u}\in\mathcal{U}$, and scalar multiplication $(\alpha\mathbb{T})\mathbf{u} := \alpha\mathbb{T}\mathbf{u}$ with $\alpha\in\mathcal{F}$.

- The linear space $\mathcal{G}(\mathcal{U},\mathcal{Z})$ is a normed space, whose norm is the usual operator norm $\|\mathbb{T}\|$ for all $\mathbb{T}\in\mathcal{G}(\mathcal{U},\mathcal{Z})$.
- Let $\mathcal{B}$ ba a Banach space; then, $\mathcal{G}(\mathcal{U},\mathcal{B})$ is a Banach space.
- Let $\mathcal{H}$ is a Hilbert space, then $\mathcal{G}(\mathcal{H},\mathcal{H})$ is a Banach algebra.

Convergence [1, §4.9]. Let $\mathcal{U}$ and $\mathcal{Z}$ be normed spaces. A sequence $\{\mathbb{T}_n\}$ of operators $\mathbb{T}_n\in\mathcal{G}(\mathcal{U},\mathcal{Z})$ is said to be

- *uniformly operator convergent* if $\{\mathbb{T}_n\}$ converges in the operator norm on $\mathcal{G}(\mathcal{U},\mathcal{Z})$, i.e., $\|\mathbb{T}_n-\mathbb{T}\|\to 0$;
- *strongly operator convergent* if $\{\mathbb{T}_n\mathbf{u}\}$ converges strongly in $\mathcal{Z}$ for every $\mathbf{u}\in\mathcal{U}$, i.e., $\|\mathbb{T}_n\mathbf{u}-\mathbb{T}\mathbf{u}\|\to 0$ for all $\mathbf{u}\in\mathcal{U}$;
- *weakly operator convergent* if $\{\mathbb{T}_n\mathbf{u}\}$ converges weakly in $\mathcal{Z}$ for every $\mathbf{u}\in\mathcal{U}$, i.e., $|f(\mathbb{T}_n\mathbf{u})-f(\mathbb{T}\mathbf{u})|\to 0$ for all $\mathbf{u}\in\mathcal{U}$ and all bounded linear functionals f on $\mathcal{U}$, that is, for all f in the dual space $\mathcal{U}'$ of $\mathcal{U}$.

Uniform convergence implies strong convergence, which in turn implies weak convergence, all with the same limit.

- Let $\mathbb{T}_n\in\mathcal{G}(\mathcal{B},\mathcal{U})$, where $\mathcal{B}$ is a Banach space and $\mathcal{U}$ a normed space. If $\{\mathbb{T}_n\}$ is strongly operator convergent with limit $\mathbb{T}$, then $\mathbb{T}\in\mathcal{G}(\mathcal{B},\mathcal{U})$.
- A sequence $\{\mathbb{T}_n\}$ of operators in $\mathcal{G}(\mathcal{B},\mathcal{Z})$, where $\mathcal{B}$ and $\mathcal{Z}$ are Banach spaces, is strongly operator convergent iff (i) the sequence $\{\|\mathbb{T}_n\|\}$ is bounded, and (ii) the sequence $\{\mathbb{T}_n\mathbf{b}\}$ is Cauchy in $\mathcal{Z}$ for every $\mathbf{b}$ in a total subset of $\mathcal{B}$.

Adjoint Operator [1, §4.5]. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let $\mathbb{T}:\mathcal{X}\to\mathcal{Y}$ be a bounded linear operator. Then, for any bounded linear functionals $f\in\mathcal{X}'$ and $g\in\mathcal{Y}'$, the *adjoint operator* $\mathbb{T}^\times:\mathcal{Y}'\to\mathcal{X}'$ of $\mathbb{T}$ is defined by $f(\mathbf{x}) = (\mathbb{T}^\times g)(\mathbf{x}) = g(\mathbb{T}\mathbf{x})$ for all $\mathbf{x}\in\mathcal{X}$.

- The adjoint operator $\mathbb{T}^\times$ is linear and bounded, and $\|\mathbb{T}^\times\| = \|\mathbb{T}\|$.
- If $\mathbb{T}$ is represented by a matrix $\mathbf{T}$, then the adjoint operator $\mathbb{T}^\times$ is represented by $\mathbf{T}^T$.
- Let $\mathbb{S}:\mathcal{X}\to\mathcal{Y}$ be another bounded linear operator. Then
  - $(\mathbb{S}+\mathbb{T})^\times = \mathbb{S}^\times + \mathbb{T}^\times$.
  - $(\alpha\mathbb{T})^\times = \alpha\mathbb{T}^\times, \quad \alpha\in\mathcal{F}$.
  - $(\mathbb{S}\mathbb{T})^\times = \mathbb{T}^\times\mathbb{S}^\times$.
- If $\mathbb{T}^{-1}$ exists and $\mathbb{T}^{-1}\in\mathcal{B}(\mathcal{X},\mathcal{Y})$, then $(\mathbb{T}^\times)^{-1}$ also exists, $(\mathbb{T}^\times)^{-1}\in\mathcal{B}(\mathcal{X}',\mathcal{Y}')$, and $(\mathbb{T}^\times)^{-1} = (\mathbb{T}^{-1})^\times$.

Closed Linear Operators [1, §4.13]. Let $\mathcal{U}$ and $\mathcal{Z}$ be normed spaces and let $\mathbb{T}:\mathcal{D}(\mathbb{T})\to\mathcal{Z}$ be a linear operator with domain $\mathcal{D}(\mathbb{T})\subset\mathcal{U}$. Then, $\mathbb{T}$ is called a *closed linear operator* if its *graph* $\mathcal{G}(\mathbb{T}) := \{(\mathbf{u},\mathbf{z}) : \mathbf{u}\in\mathcal{D}(\mathbb{T}), \mathbf{z} = \mathbb{T}\mathbf{u}\}$ is closed in the normed space $\mathcal{U}\times\mathcal{Z}$.

- *Closed graph theorem:* Let $\mathbb{T}$ be a closed operator. If $\mathcal{D}(\mathbb{T})$ is closed in $\mathcal{V}$, the operator $\mathbb{T}$ is bounded.
- Let $\mathbb{T}:\mathcal{D}(\mathbb{T})\to\mathcal{Z}$ be a linear operator, where $\mathcal{D}(\mathbb{T})\subset\mathcal{U}$ and $\mathcal{U},\mathcal{Z}$ are normed spaces. Then, $\mathbb{T}$ is closed iff it has the following property: If $\mathbf{u}_n\to\mathbf{u}$ for $\mathbf{u}_n\in\mathcal{D}(\mathbb{T})$, and $\mathbb{T}\mathbf{u}_n\to\mathbf{z}$, then $\mathbf{u}\in\mathcal{D}(\mathbb{T})$ and $\mathbb{T}\mathbf{u} = \mathbf{z}$.

Compact Linear Operators [1, 2]. Let $\mathcal{U}$ and $\mathcal{Z}$ be normed spaces. A linear operator $\mathbb{T}:\mathcal{U}\to\mathcal{Z}$ is called *compact* or *completely continuous* if for every bounded subset $\mathcal{S}\subset\mathcal{U}$, the image $\mathbb{T}(\mathcal{S})$ is *relatively compact*, i.e., the closure $\overline{\mathbb{T}(\mathcal{S})}$ is (sequentially) compact.

- Every compact linear operator $\mathbb{T}$ is bounded and, therefore, continuous.
- If $\dim\mathcal{U} = \infty$, the identity operator $\mathbb{I}$, which is continuous, is not compact.
- A linear operator $\mathbb{T}:\mathcal{U}\to\mathcal{Z}$ is compact iff it maps every bounded sequence $\{\mathbf{u}_n\}$ in $\mathcal{U}$ onto a sequence $\{\mathbb{T}\mathbf{u}_n\}$ in $\mathcal{Z}$ that has a convergent subsequence.
- If $\mathbb{T}$ is bounded and $\dim\mathcal{R}(\mathbb{T})<\infty$, then $\mathbb{T}$ is compact.
- If $\mathcal{U}$ is a finite-dimensional normed linear space, every linear operator defined on $\mathcal{U}$ is compact.
- Given $\epsilon>0$, there exists a finite-dimensional subspace $\mathcal{M}\subset\mathcal{R}(\mathbb{T})$ such that
$$\inf_{\mathbf{m}\in\mathcal{M}}\|\mathbb{T}\mathbf{u}-\mathbf{m}\| < \epsilon\|\mathbf{u}\|$$
for any $\mathbf{u}\in\mathcal{U}$.
- Let $\{\mathbf{u}_n\}$ be a weakly convergent sequence in $\mathcal{U}$ with $\mathbf{u}_n\xrightarrow{w}\mathbf{u}$. Then $\{\mathbb{T}\mathbf{u}_n\}$ is strongly convergent in $\mathcal{Z}$ and has the strong limit $\mathbf{z} = \mathbb{T}\mathbf{u}$.
- If $\mathbb{T}$ is compact, so is its adjoint operator $\mathbb{T}^\times:\mathcal{Z}'\to\mathcal{U}'$.
- Let $\{\mathbb{T}_n\}$ be a sequence of compact linear operators from a normed space $\mathcal{U}$ into a Banach space $\mathcal{B}$. If $\{\mathbb{T}_n\}$ is uniformly operator convergent, i.e., $\|\mathbb{T}_n-\mathbb{T}\|\to 0$, then the limit operator $\mathbb{T}$ is compact.
- A compact linear operator $\mathbb{T}:\mathcal{U}\to\mathcal{B}$ from a normed space $\mathcal{U}$ into a Banach space $\mathcal{B}$ has a compact linear extension $\tilde{\mathbb{T}}:\tilde{\mathcal{U}}\to\mathcal{B}$, where $\tilde{\mathcal{U}}$ is the completion of $\mathcal{U}$.
- Let $\mathbb{T}:\mathcal{B}\to\mathcal{A}$ and $\mathbb{S}:\mathcal{B}\to\mathcal{A}$ be compact linear operators, where $\mathcal{B}$ and $\mathcal{A}$ are Banach spaces. Then, $\mathbb{T}+\mathbb{S}$ is compact.
- Let $\mathbb{T}:\mathcal{U}\to\mathcal{U}$ be a compact linear operator and $\mathbb{S}:\mathcal{U}\to\mathcal{U}$ a bounded linear operator on a normed space $\mathcal{U}$. Then $\mathbb{T}\mathbb{S}$ and $\mathbb{S}\mathbb{T}$ are compact.

Spectral Theory of Linear Operators

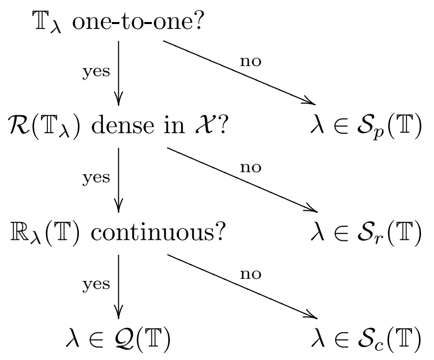
Resolvent, Spectrum [2, 1]. Let $\mathcal{B}$ be a complex Banach space, and let $\mathbb{T}:\mathcal{D}(\mathbb{T})\to\mathcal{R}(\mathbb{T})$ be a linear operator with $\mathcal{D}(\mathbb{T}),\mathcal{R}(\mathbb{T})\subset\mathcal{B}$.

- Associated with $\mathbb{T}$ is the the operator $\mathbb{T}_\lambda := \mathbb{T}-\lambda\mathbb{I}$, where $\lambda\in\mathbb{C}$ and $\mathbb{I}$ denotes the identity operator.
- If $\mathbb{T}_\lambda$ has an inverse defined on its range, it is called the *resolvent* of $\mathbb{T}$ and denoted as $\mathbb{R}_\lambda(\mathbb{T}) := \mathbb{T}_\lambda^{-1} = (\mathbb{T}-\lambda\mathbb{I})^{-1}$ on $\mathcal{R}(\mathbb{T}_\lambda)$.

The *resolvent set* $\mathcal{Q}(\mathbb{T})$ of $\mathbb{T}$ is defined as the set of all complex numbers λ such that the range of $\mathbb{T}_\lambda$ is dense in $\mathcal{B}$ and that $\mathbb{T}_\lambda$ has a continuous inverse defined on its range. The numbers $\lambda\in\mathcal{Q}(\mathbb{T})$ are called *regular values*. The set $\mathcal{S}(\mathbb{T}) := \mathcal{Q}(\mathbb{T})^c$ is called the *spectrum* of $\mathbb{T}$; a $\lambda\in\mathcal{S}(\mathbb{T})$ is called a *spectral value* of $\mathbb{T}$. The spectrum $\mathcal{S}(\mathbb{T})$ can be partitioned into three disjoint sets:

- The *point spectrum* $\mathcal{S}_p(\mathbb{T})$ is the set such that $\mathbb{T}_\lambda$ is not one-to-one. A $\lambda\in\mathcal{S}_p(\mathbb{T})$ is called an *eigenvalue* of $\mathbb{T}$.
- The *continuous spectrum* $\mathcal{S}_c(\mathbb{T})$ is the set such that $\mathbb{T}_\lambda$ is one-to-one, has its range dense set in $\mathcal{B}$, but $\mathbb{R}_\lambda(\mathbb{T})$, defined on $\mathcal{R}(\mathbb{T}_\lambda)$, is not continuous and, therefore, unbounded.
- The *residual spectrum* $\mathcal{S}_r(\mathbb{T})$ is the set such that $\mathbb{T}$ is one-to-one, but $\mathcal{R}(\mathbb{T}_\lambda)$ is not dense in $\mathcal{B}$.

In summary:



- The four sets are pairwise disjoint and $\mathbb{C} = \mathcal{Q}(\mathbb{T})\cup\mathcal{S}_p(\mathbb{T})\cup\mathcal{S}_c(\mathbb{T})\cup\mathcal{S}_r(\mathbb{T})$; some of the sets may be empty.
- If $\mathbb{R}_\lambda(\mathbb{T})$ exists, it is a linear operator.
- Let $\mathcal{B}$ be a complex Banach space, $\mathbb{T}:\mathcal{B}\to\mathcal{B}$ a linear operator, and $\lambda\in\mathcal{Q}(\mathbb{T})$. If $\mathbb{T}$ is closed or bounded, then, $\mathbb{R}_\lambda(\mathbb{T})$ is defined on the whole space $\mathcal{B}$ and is bounded.

Eigenvalues [1, §7]. Let $\mathcal{U}$ be a normed space over the complex field and $\mathbb{T}:\mathcal{D}(\mathbb{T})\to\mathcal{U}$ a linear operator with domain $\mathcal{D}(\mathbb{T})\subset\mathcal{U}$.

- The resolvent $\mathbb{R}_\lambda(\mathbb{T})$ exists iff $\mathbb{T}\mathbf{u} = \mathbf{0}$ implies $\mathbf{u} = \mathbf{0}$, i.e., the null space $\mathcal{N}(\mathbb{T})$ is $\{\mathbf{0}\}$.
- If $\mathbb{T}_\lambda\mathbf{u} = \mathbf{0}$ for some $\mathbf{u}\neq\mathbf{0}$, then $\lambda\in\mathcal{S}_p(\mathbb{T})$. The vector $\mathbf{u}$ is then called an *eigenvector* of $\mathbb{T}$ with eigenvalue λ.
- The subspace of $\mathcal{D}(\mathbb{T})$ that consists of $\mathbf{0}$ and all eigenvectors of $\mathbb{T}$ with eigenvalue λ is called the *eigenspace* of $\mathbb{T}$ corresponding to that eigenvalue.
- Eigenvectors with different eigenvalues constitute a linearly independent set.

Spectral Properties of Operators on Normed Spaces

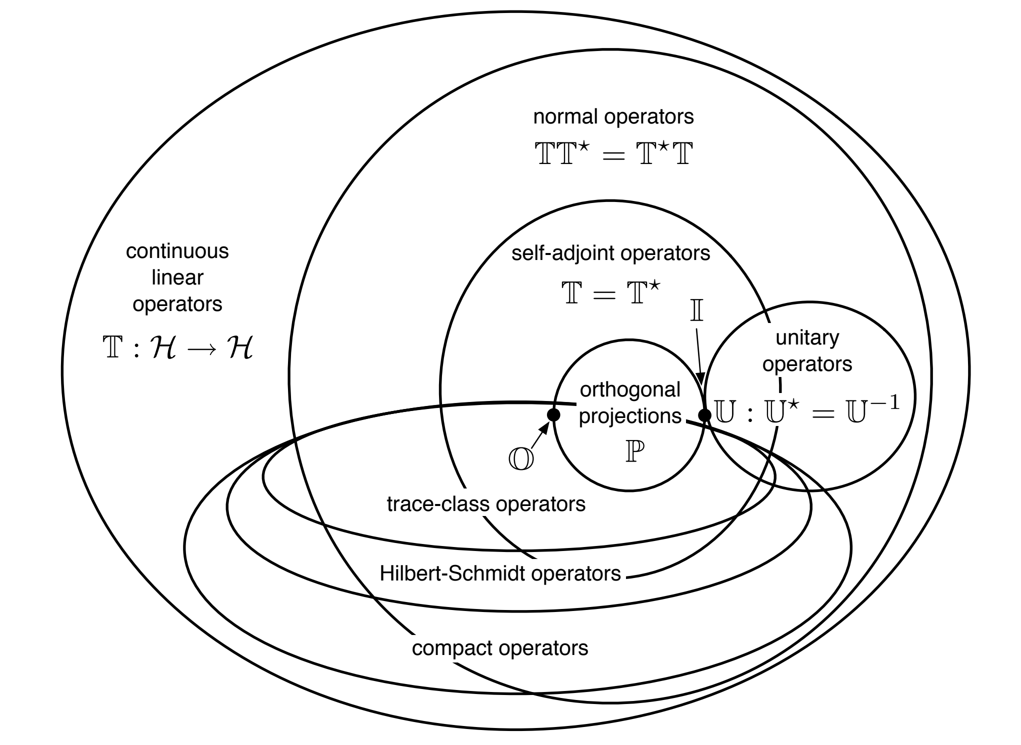
Bounded Linear Operators on a Complex Banach Space [1, §7.3]. Let $\mathcal{B}$ be a complex Banach space, and let $\mathbb{T}\in\mathcal{G}(\mathcal{B},\mathcal{B})$ be a bounded linear operator.

- The resolvent set $\mathcal{Q}(\mathbb{T})$ is not empty.
- The spectrum $\mathcal{S}(\mathbb{T})$ is not empty.
- The resolvent set $\mathcal{Q}(\mathbb{T})$ is open; hence, the spectrum $\mathcal{S}(\mathbb{T})$ is closed.
- If $\|\mathbb{T}\|<1$, then $(\mathbb{I}-\mathbb{T})^{-1}$ exists, is a bounded linear operator on the whole space $\mathcal{B}$, and has the following series expansion, convergent in the norm on $\mathcal{G}(\mathcal{B},\mathcal{B})$:
$$(\mathbb{I}-\mathbb{T})^{-1} = \sum_{n=0}^{\infty}\mathbb{T}^n = \mathbb{I}+\mathbb{T}+\mathbb{T}^2+\dots$$
and $\|(\mathbb{I}-\mathbb{T})^{-1}\|\leq(1-\|\mathbb{T}\|)^{-1}$.
- For every $\lambda_0\in\mathcal{Q}(\mathbb{T})$, the resolvent $\mathbb{R}_\lambda(\mathbb{T})$ has the representation
$$\mathbb{R}_\lambda(\mathbb{T}) = \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n\mathbb{R}_{\lambda_0}^{n+1}.$$
- The resolvent $\mathbb{R}_\lambda(\mathbb{T})$ is holomorphic at every point λ_0 of the resolvent set $\mathcal{Q}(\mathbb{T})$. Hence, it is locally holomorphic on $\mathcal{Q}(\mathbb{T})$.
- The *spectral radius* of $\mathbb{T}$ is defined as $r_{\mathbb{T}} := \sup_{\lambda\in\mathcal{S}(\mathbb{T})}|\lambda|$.
- The spectral radius is given as $r_{\mathbb{T}} = \lim_{n\to\infty}\|\mathbb{T}^n\|^{1/n}$.
- The spectrum $\mathcal{S}(\mathbb{T})$ is compact and lies in a disk with spectral radius $r_{\mathbb{T}}\leq\|\mathbb{T}\|$.
- Let $\lambda,\mu\in\mathbb{R}_\lambda(\mathbb{T})$. Then,
  - The resolvent $\mathbb{R}_\lambda(\mathbb{T})$ satisfies the *Hilbert relation*, also called *resolvent identity:*
  $$\mathbb{R}_\mu-\mathbb{R}_\lambda = (\mu-\lambda)\mathbb{R}_\mu\mathbb{R}_\lambda;$$
  - $\mathbb{R}_\lambda(\mathbb{T})$ commutes with any $\mathbb{S}\in\mathcal{G}(\mathcal{B},\mathcal{B})$ that commutes with $\mathbb{T}$;
  - $\mathbb{R}_\lambda\mathbb{R}_\mu = \mathbb{R}_\mu\mathbb{R}_\lambda$.
- *Spectral mapping:* Let $p(\lambda) := \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1}+\dots+\alpha_0\lambda^0$ with $\alpha_n\neq 0$. Then, $\mathcal{S}(p(\mathbb{T})) = p(\mathcal{S}(\mathbb{T}))$. That is, the spectrum of the operator $p(\mathbb{T}) = \alpha_n\mathbb{T}^n + \alpha_{n-1}\mathbb{T}^{n-1}+\dots+\alpha_0\mathbb{I}$ consists of all those values that the polynomial p assumes on the spectrum $\mathcal{S}(\mathbb{T})$ of $\mathbb{T}$.

Compact Linear Operators [1, §8]. Let $\mathbb{T}:\mathcal{U}\to\mathcal{U}$ be a compact operator on a normed space $\mathcal{U}$, and let $\mathbb{T}_\lambda := \mathbb{T}-\lambda\mathbb{I}$.

- Every spectral value $\lambda\in\mathcal{S}(\mathbb{T}), \lambda\neq 0$, if it exists, is an eigenvalue of $\mathbb{T}$.
- The set of eigenvalues $\mathcal{S}_p(\mathbb{T})$ is at most countable, and its only possible limit point is $\lambda=0$.
- If $\lambda=0\in\mathcal{Q}(\mathbb{T})$, then $\mathbb{T}$ is finite dimensional.
- For every $\lambda\neq 0$ and every $n=1,2,\ldots$, the null space $\mathcal{N}(\mathbb{T}^n_\lambda)$ is finite dimensional and the range $\mathcal{R}(\mathbb{T}^n_\lambda)$ is closed.
- There exists a smallest integer $n=r$, depending on λ, such that
$$\mathcal{N}(\mathbb{T}^r_\lambda)=\mathcal{N}(\mathbb{T}^{r+1}_\lambda)=\mathcal{N}(\mathbb{T}^{r+2}_\lambda)\ldots$$
and
$$\mathbb{T}^r_\lambda(\mathcal{U})=\mathbb{T}^{r+1}_\lambda(\mathcal{U})=\mathbb{T}^{r+2}_\lambda(\mathcal{U})\ldots$$
If $r>0$, the inclusions
$$\mathcal{N}(\mathbb{T}^0_\lambda)\subset\mathcal{N}(\mathbb{T}^1_\lambda)\subset\cdots\subset\mathcal{N}(\mathbb{T}^r_\lambda)$$
and
$$\mathbb{T}^0_\lambda(\mathcal{U})\supset\mathbb{T}^1_\lambda(\mathcal{U})\supset\cdots\supset\mathbb{T}^r_\lambda(\mathcal{U})$$
are proper. Furthermore, the space $\mathcal{U}$ can be represented as $\mathcal{U}=\mathcal{N}(\mathbb{T}^r_\lambda)\oplus\mathbb{T}^r_\lambda(\mathcal{U})$.

Linear Operators and Functionals on Hilbert Space



Representation of Functionals [1, §3.8].

- *Riesz Theorem:* Every bounded linear functional f on a Hilbert space $\mathcal{H}$ can be represented by an inner product $f(\mathbf{h})=\langle\mathbf{h},\mathbf{z}\rangle$, where $\mathbf{h}\in\mathcal{H}$, and where $\mathbf{z}\in\mathcal{H}$ is uniquely determined by f and has norm $\|\mathbf{z}\|=\|f\|$.
- *Riesz representation:* Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $\mathbf{h}_1\in\mathcal{H}_1,\mathbf{h}_2\in\mathcal{H}_2$, and $g:\mathcal{H}_1\times\mathcal{H}_2\to\mathcal{F}$ a bounded sesquilinear form. Then g has a representation $g(\mathbf{h}_1,\mathbf{h}_2)=\langle\mathbb{S}\mathbf{h}_1,\mathbf{h}_2\rangle$, where $\mathbb{S}:\mathcal{H}_1\to\mathcal{H}_2$ is a bounded linear operator that is uniquely determined by g and has norm $\|\mathbb{S}\|=\|g\|$.

Hilbert Adjoint Operator [1, 2]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{Z}$ be a bounded linear operator that maps the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{Z}$. The *Hilbert adjoint operator* $\mathbb{T}^\star$ of $\mathbb{T}$ is the operator $\mathbb{T}^\star:\mathcal{Z}\to\mathcal{H}$ such that $\langle\mathbb{T}\mathbf{h},\mathbf{z}\rangle=\langle\mathbf{h},\mathbb{T}^\star\mathbf{z}\rangle$ for all $\mathbf{h}\in\mathcal{H}$ and $\mathbf{z}\in\mathcal{Z}$. This operator exists, is unique, and is a bounded linear operator with norm $\|\mathbb{T}^\star\|=\|\mathbb{T}\|$.
Let $\mathbb{S}:\mathcal{H}\to\mathcal{Z}$ be another bounded linear operator, and let α be any scalar. The Hilbert adjoint operator has the following properties:

- $\mathbb{I}^\star=\mathbb{I}$.
- $\langle\mathbb{T}^\star\mathbf{h},\mathbf{z}\rangle=\langle\mathbf{z},\mathbb{T}\mathbf{h}\rangle$.
- $(\mathbb{S}+\mathbb{T})^\star=\mathbb{S}^\star+\mathbb{T}^\star$.
- $(\alpha\mathbb{T})^\star=\alpha^*\mathbb{T}^\star$.
- $(\mathbb{T}^\star)^\star=\mathbb{T}$.
- $\|\mathbb{T}^\star\mathbb{T}\|=\|\mathbb{T}\mathbb{T}^\star\|=\|\mathbb{T}\|^2$.
- $\mathbb{T}^\star\mathbb{T}=0\iff\mathbb{T}=0$.
- $(\mathbb{S}\mathbb{T})^\star=\mathbb{T}^\star\mathbb{S}^\star$.
- If $\mathbb{T}$ can be represented by a matrix $\mathbf{T}$, then $\mathbb{T}^\star$ can be represented by $\mathbf{T}^H$.

Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ a bounded linear operator that maps a Hilbert space $\mathcal{H}$ into itself.

- The ranges and null spaces of $\mathbb{T}$ and $\mathbb{T}^\star$ are related as follows:
  - $\overline{\mathcal{R}(\mathbb{T})}=\mathcal{N}^\perp(\mathbb{T}^\star)$.
  - $\mathcal{N}^\perp(\mathbb{T})=\overline{\mathcal{R}(\mathbb{T}^\star)}$.
- Let $\mathbb{T}$ be continuous, and let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Then, $\mathcal{M}$ is invariant under $\mathbb{T}$ iff $\mathcal{M}^\perp$ is invariant under $\mathbb{T}^\star$.
- A closed linear subspace $\mathcal{M}\subset\mathcal{H}$ reduces $\mathbb{T}$ iff $\mathcal{M}$ is invariant under both $\mathbb{T}$ and $\mathbb{T}^\star$.
- The Hilbert adjoint operator $\mathbb{T}^\star:\mathcal{Z}\to\mathcal{H}$ and the adjoint operator $\mathbb{T}^\times:\mathcal{Z}'\to\mathcal{H}'$ are related as $\mathbb{T}^\star=\mathbb{A}_1\mathbb{T}^\times\mathbb{A}_2^{-1}$, where $\mathbb{A}_1:\mathcal{H}'\to\mathcal{H}$ and $\mathbb{A}_2:\mathcal{Z}'\to\mathcal{Z}$ are bijective, isometric, conjugate linear operators that are uniquely defined by Riesz's theorem.

A bounded linear operator $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ on a Hilbert space $\mathcal{H}$ is said to be

- *normal*, if $\mathbb{T}\mathbb{T}^\star=\mathbb{T}^\star\mathbb{T}$,
- *unitary*, if $\mathbb{T}$ is bijective and if $\mathbb{T}^\star=\mathbb{T}^{-1}$,
- *self adjoint* or *Hermitian*, if $\mathbb{T}^\star=\mathbb{T}$.

If $\mathbb{T}$ is self adjoint or unitary, it is normal.

Unitary Operators. Let the operators $\mathbb{U},\mathbb{V}:\mathcal{H}\to\mathcal{H}$ be unitary, $\mathcal{H}$ a Hilbert space. Then,

- $\mathbb{U}$ is isometric, i.e., $\|\mathbb{U}\mathbf{h}\|=\|\mathbf{h}\|$ for all $\mathbf{h}\in\mathcal{H}$,
- $\|\mathbb{U}\|=1$,
- $\mathbb{U}^{-1}$ is unitary,
- $\mathbb{U}\mathbb{V}$ is unitary.
- A bounded linear operator on a Hilbert space over the complex field is unitary iff it is isometric and onto.

Polar Decomposition [8, §30]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ be a compact linear operator on a separable complex Hilbert space $\mathcal{H}$; let $\mathbb{T}^\star$ denote the Hilbert adjoint of $\mathbb{T}$.

- The operator $\mathbb{T}$ can be factored as $\mathbb{T}=\mathbb{U}\mathbb{A}$, where $\mathbb{A}$ is a positive Hermitian operator and $\mathbb{U}^\star\mathbb{U}=\mathbb{I}$ on the range of $\mathbb{A}$. The above factorization is called the *polar decomposition* of $\mathbb{T}$; the operator $\mathbb{A}$ is called the *absolute value* of $\mathbb{T}$. The polar decomposition exists even if $\mathbb{T}$ is bounded instead of compact.
- The absolute value $\mathbb{A}$ can be taken as $\mathbb{A}:=(\mathbb{T}^\star\mathbb{T})^{1/2}$, the unique positive square root of $\mathbb{T}^\star\mathbb{T}$; the operator $\mathbb{U}$ satisfies $\mathbb{U}:\mathbb{A}\mathbf{h}\to\mathbb{T}\mathbf{h}$ for all $\mathbf{h}\in\mathcal{H}$.
- If $\mathbb{T}$ is compact, then its absolute value $\mathbb{A}$ is compact.

Singular Values [8, §30]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ be a compact linear operator on a separable complex Hilbert space $\mathcal{H}$, and let $\mathbb{T}^\star$ be its Hilbert adjoint. Furthermore, let $\mathbb{T}=\mathbb{U}\mathbb{A}$ be the polar decomposition of $\mathbb{T}$, and let $\{\sigma_n\}$ denote the set of nonzero eigenvalues of $\mathbb{A}$; they are all positive, as $\mathbb{A}$ is Hermitian. Let the σ_n be indexed in decreasing order. The numbers σ_n are called the *singular values* of $\mathbb{T}$, denoted also as $\sigma_n(\mathbb{T})$.

- The singular values of $\mathbb{T}$ form an at most countable sequence whose only possible limit point is 0.
- Let the nonzero eigenvalues of $\mathbb{T}$ be $\lambda_1(\mathbb{T}),\lambda_2(\mathbb{T}),\ldots$, arranged in decreasing order of their absolute value, including multiplicities. Then, for any $N\in\mathbb{Z}_+$
$$\prod_{n=1}^{N}|\lambda_n(\mathbb{T})|\leq\prod_{n=1}^{N}\sigma_n(\mathbb{T}).$$

Normal Operators [2]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ be a normal operator of a Hilbert space $\mathcal{H}$ into itself.

- Let $\mathbf{e}_n$ be an eigenvector associated with the eigenvalue λ_n. Then, the vector $\mathbf{e}_n$ is also an eigenvector of the Hilbert adjoint operator $\mathbb{T}^\star$ of $\mathbb{T}$ and associated with the eigenvalue λ_n^*.
- The null space satisfies $\mathcal{N}(\mathbb{T}-\lambda\mathbb{I})=\mathcal{N}(\mathbb{T}^\star-\lambda^*\mathbb{I})$.
- For any $\mu\neq\lambda$, the null spaces $\mathcal{N}(\mathbb{T}-\lambda\mathbb{I})$ and $\mathcal{N}(\mathbb{T}-\mu\mathbb{I})$ are orthogonal to one another.
- For each complex number λ, the closed linear subspace $\mathcal{N}(\mathbb{T}_\lambda-\lambda\mathbb{I})$ reduces $\mathbb{T}$.
- $\|\mathbb{T}^2\|=\|\mathbb{T}\|^2$.
- A bounded linear operator $\mathbb{T}$ on a Hilbert space $\mathcal{H}$ is normal iff $\|\mathbb{T}^\star\mathbf{h}\|=\|\mathbb{T}\mathbf{h}\|$ for every $\mathbf{h}\in\mathcal{H}$.
- The residual spectrum $\mathcal{S}_r(\mathbb{T})$ of a normal operator is empty.
- A complex number λ is in $\mathcal{S}(\mathbb{T})$ iff there exists a sequence $\{\mathbf{h}_n\}$ with $\mathbf{h}_n\in\mathcal{H}$, $\|\mathbf{h}_n\|=1$ for all n, such that $\|(\mathbb{T}-\lambda\mathbb{I})\mathbf{h}_n\|\to 0$ as $n\to\infty$; in other words, the operator $\mathbb{T}-\lambda\mathbb{I}$ is *not* bounded below.
- Let a bounded linear operator $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ have the Cartesian decomposition $\mathbb{H}=\mathbb{T}+i\mathbb{S}$, where $\mathbb{T}$ and $\mathbb{S}$ are self-adjoint. Then, $\mathbb{H}$ is normal iff $\mathbb{T}$ and $\mathbb{S}$ commute. In that case, $\max\{\|\mathbb{T}\|^2,\|\mathbb{S}\|^2\}\leq\|\mathbb{H}\|^2\leq\|\mathbb{T}\|^2+\|\mathbb{S}\|^2$.
- Let $\mathbb{T}$ and $\mathbb{S}$ be normal operators on a Hilbert space $\mathcal{H}$ such that one commutes with the adjoint of the other, i.e., $\mathbb{T}\mathbb{S}^\star=\mathbb{S}^\star\mathbb{T}$ and $\mathbb{T}^\star\mathbb{S}=\mathbb{S}\mathbb{T}^\star$, or such that the two operators commute, i.e., $\mathbb{T}\mathbb{S}=\mathbb{S}\mathbb{T}$; then, $\mathbb{T}+\mathbb{S}$, $\mathbb{T}\mathbb{S}$, and $\mathbb{S}\mathbb{T}$ are normal.

Bounded Self-Adjoint Linear Operators [1, 2]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ be a bounded self-adjoint linear operator on a complex Hilbert space $\mathcal{H}$, let $\mathbb{T}_\lambda:=\mathbb{T}-\lambda\mathbb{I}$, and let $\mathbf{h}\in\mathcal{H}$.

- The set of all self-adjoint linear operators on $\mathcal{H}$ is a closed set in $\mathcal{G}(\mathcal{H},\mathcal{H})$.
- The set of all self-adjoint linear operators on $\mathcal{H}$ forms a *real* normed linear space under the operator norm.
- A bounded linear operator $\mathbb{T}$ on a complex Hilbert space $\mathcal{H}$ is self adjoint iff $\langle\mathbb{T}\mathbf{h},\mathbf{h}\rangle=\langle\mathbf{h},\mathbb{T}\mathbf{h}\rangle$ is real for all $\mathbf{h}\in\mathcal{H}$. If $\mathcal{H}$ is a real Hilbert space, the direct part holds but the converse is no longer true.
- The spectrum $\mathcal{S}(\mathbb{T})$ of $\mathbb{T}$ lies in the closed intervall $[m_\mathbb{T},M_\mathbb{T}]\in\mathbb{R}$, where
$$m_\mathbb{T}=\inf_{\|\mathbf{h}\|}\langle\mathbb{T}\mathbf{h},\mathbf{h}\rangle,\qquad M_\mathbb{T}=\sup_{\|\mathbf{h}\|}\langle\mathbb{T}\mathbf{h},\mathbf{h}\rangle.$$
Both $m_\mathbb{T}$ and $M_\mathbb{T}$ are spectral values of $\mathbb{T}$.
- The operator norm of $\mathbb{T}$ is given by
$$\|\mathbb{T}\|=\max(|m_\mathbb{T}|,|M_\mathbb{T}|)=\sup_{\|\mathbf{h}\|}|\langle\mathbb{T}\mathbf{h},\mathbf{h}\rangle|.$$
- Eigenvectors that correspond to numerically different eigenvalues of $\mathbb{T}$ are orthogonal
- A number λ belongs to the resolvent set $\mathbb{R}_\lambda(\mathbb{T})$ iff there exists a $c>0$ such that $\|\mathbb{T}_\lambda\mathbf{h}\|\geq c\|\mathbf{h}\|$ for every $\mathbf{h}\in\mathcal{H}$.
- The product of two self adjoint linear operators on a Hilbert space is self adjoint only if the operators commute.
- Every bounded linear operator $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ has a so-called *Cartesian decomposition*: $\mathbb{T}=\mathbb{A}+i\mathbb{B}$, where $\mathbb{A}$ and $\mathbb{B}$ are self-adjoint.
  - The Cartesian decomposition is unique.
  - $\mathbb{A}=1/2(\mathbb{T}+\mathbb{T}^\star)$.
  - $\mathbb{B}=1/(2i)(\mathbb{T}-\mathbb{T}^\star)$.
  - If λ is an eigenvalue of $\mathbb{T}$, then $\lambda=\alpha+i\beta$, where α is an eigenvalue of $\mathbb{A}$ and β is an eigenvalue of $\mathbb{B}$.

Nonnegative Self-Adjoint Linear Operators [1, §9]. Consider the set of all bounded, self-adjoint, linear operators on a complex Hilbert space $\mathcal{H}$. A reflexive partial ordering $\preceq$ on this set is defined by $\mathbb{T}_1\preceq\mathbb{T}_2$ iff $\langle\mathbb{T}_1\mathbf{h},\mathbf{h}\rangle\leq\langle\mathbb{T}_2\mathbf{h},\mathbf{h}\rangle$ for $\mathbf{h}\in\mathcal{H}$. A bounded, self-adjoint, linear operator $\mathbb{T}$ is said to be *nonnegative* (although not strictly correct, sometimes also called *positive*) and denoted $\mathbb{T}\succeq\mathbb{O}$, if $\langle\mathbb{T}\mathbf{h},\mathbf{h}\rangle\geq 0$ for all $\mathbf{h}\in\mathcal{H}$.

- $\mathbb{T}_1\preceq\mathbb{T}_2\iff\mathbb{O}\preceq\mathbb{T}_2-\mathbb{T}_1$.
- If two bounded, self-adjoint, linear operators $\mathbb{T}$ and $\mathbb{S}$ are nonnegative and commute, i.e., $\mathbb{T}\mathbb{S}=\mathbb{S}\mathbb{T}$, then their product $\mathbb{S}\mathbb{T}$ is nonnegative.
- If $\mathbb{T}$ is bounded and self adjoint, then $\mathbb{T}^2$ is nonnegative.

A *monotone sequence* $\{\mathbb{T}_n\}$ of bounded, self-adjoint, linear operators is a sequence that is either *monotonically increasing*, i.e., $\mathbb{T}_1\preceq\mathbb{T}_2\preceq\mathbb{T}_3\preceq\ldots$, or *monotonically decreasing*, $\mathbb{T}_1\succeq\mathbb{T}_2\succeq\mathbb{T}_3\succeq\ldots$.

- Let $\{\mathbb{T}_n\}$ be a monotonically increasing sequence of bounded, self-adjoint, linear operators such that $\mathbb{T}_1\preceq\mathbb{T}_2\preceq\ldots\preceq\mathbb{T}_n\preceq\ldots\preceq\mathbb{S}$, where $\mathbb{S}$ is also bounded and self adjoint. Suppose that all elements of the sequence commute pairwise and also commute with $\mathbb{S}$. Then, $\{\mathbb{T}_n\}$ is strongly operator convergent, $\mathbb{T}_n\mathbf{h}\to\mathbb{T}\mathbf{h}$ for all $\mathbf{h}\in\mathcal{H}$, and the limit operator $\mathbb{T}$ is linear, bounded, self adjoint, and satisfies $\mathbb{T}\preceq\mathbb{S}$.

A bounded, self-adjoint linear operator $\mathbb{S}$ is called a *square root* of another bounded, self-adjoint, linear operator $\mathbb{T}$ if $\mathbb{S}^2=\mathbb{T}$. If, in addition, $\mathbb{S}\succeq\mathbb{O}$, then $\mathbb{S}$ is called a *nonnegative square root* of $\mathbb{T}$ and is denoted by $\mathbb{S}=\mathbb{T}^{1/2}$.

- Every nonnegative, bounded, self-adjoint, linear operator $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ on a complex Hilbert space $\mathcal{H}$ has a nonnegative square root $\mathbb{S}$ that is unique.
- The square-root operator $\mathbb{S}$ of $\mathbb{T}$ commutes with every bounded linear operator on $\mathcal{H}$ that commutes with $\mathbb{T}$.

Compact Normal Operators [2]. Let $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ be a normal operator on a nontrivial Hilbert space $\mathcal{H}$, and let $\mathbb{T}$ have the Cartesian decomposition $\mathbb{T}=\mathbb{A}+i\mathbb{B}$.

- The operator $\mathbb{T}$ is compact iff both $\mathbb{A}$ and $\mathbb{B}$ are compact.
- The operator $\mathbb{T}$ is compact iff $\mathbb{T}^\star$ is compact.
- If $\mathbb{T}$ is compact, it has an eigenvalue λ with $\max\{\|\mathbb{A}\|,\|\mathbb{B}\|\}\leq|\lambda|$. If $\mathbb{T}$ is self-adjoint, then it has an eigenvalue λ with $\lambda=\|\mathbb{T}\|$.
- If $\mathbb{T}$ is compact and has no eigenvalues, then $\mathcal{H}=\{\mathbf{0}\}$.
- If $\mathcal{H}$ is not separable, then $\lambda=0$ is necessarily an eigenvalue of any compact normal operator on $\mathcal{H}$.

Hilbert-Schmidt Operators [9, 2, 8]. Let $\{\mathbf{x}_n\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. A bounded linear operator $\mathbb{T}:\mathcal{H}\to\mathcal{H}$ is called a *Hilbert-Schmidt(HS)* operator if $\sum_{n=1}^{\infty}\|\mathbb{T}\mathbf{x}_n\|^2<\infty$. The number
$$\|\mathbb{T}\|_{\mathrm{HS}}:=\left(\sum_{n=1}^{\infty}\|\mathbb{T}\mathbf{x}_n\|^2\right)^{1/2}$$
is called the *Hilbert-Schmidt norm* of $\mathbb{T}$.

- The HS norm does not depend on the choice of orthonormal basis for $\mathcal{H}$.
- The HS norm of a matrix is also called the *Frobenius norm*.
- If $\mathbb{T}$ is HS, then $\mathbb{T}^\star$ is HS, and $\|\mathbb{T}\|\leq\|\mathbb{T}\|_{\mathrm{HS}}$, as well as $\|\mathbb{T}\|_{\mathrm{HS}}=\|\mathbb{T}^\star\|_{\mathrm{HS}}$.
- Every HS operator is compact; hence it is bounded and continuous.
- Every HS operator is the limit in HS-norm of a sequence of operators with finite-dimensional range.
- A compact linear operator is HS iff $\sum_n\sigma_n^2(\mathbb{T})<\infty$.
- For a representation of a given Hilbert space as $\mathcal{L}^2(\mathcal{H},\mathscr{M},\mu)$ with positive measure μ and the corresponding collection $\mathscr{M}$ of measurable subsets, HS operators are those operators $\mathbb{T}$ that have a representation in the form
$$(\mathbb{T}\mathbf{f})(t)=\int_{\mathcal{H}}k(t,s)f(s)d\mu(s),$$
where $\mathbf{f}=f(t)\in\mathcal{L}^2(\mathcal{H},\mathscr{M},\mu)$, and the integral *kernel* $k(t,s)$ satisfies
$$\iint_{\mathcal{H}}|k(t,s)|^2\,d\mu(s)d\mu(t)<\infty.$$
- If $\mathbb{T}\in\mathcal{S}(\mathcal{H})$, and f is a single-valued analytic function on $\mathcal{S}(\mathbb{T})$ that vanishes at zero, then $f(\mathbb{T})$ is a HS operator, and the mapping $\mathbb{T}\to f(\mathbb{T})$ of $\mathcal{S}(\mathcal{H})$ into itsef is continuous.

The set $\mathcal{S}(\mathcal{H})$ of all HS operators on a Hilbert space $\mathcal{H}$, together with the HS norm, is a Banach algebra with $\|\mathbb{T}\mathbb{S}\|_{\mathrm{HS}}\leq\|\mathbb{T}\|_{\mathrm{HS}}\cdot\|\mathbb{S}\|_{\mathrm{HS}}$ for every $\mathbb{T},\mathbb{S}\in\mathcal{S}(\mathcal{H})$. It contains operators of finite range as a dense subset. The set of HS operators is a self-adjoint ideal in $\mathcal{G}(\mathcal{H},\mathcal{H})$, the Banach algebra of all bounded linear operators in Hilbert space.

Trace Class Operators [8, §30]. A compact linear operator $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is said to be in *trace class* if

$$\sum_{n=1}^{\infty} \sigma_n(\mathbb{T}) < \infty.$$

The above sum defines the *trace norm* $\|\mathbb{T}\|_{\rm tr}$:

$$\|\mathbb{T}\|_{\rm tr} := \sum_{n=1}^{\infty} \sigma_n(\mathbb{T}).$$

For $\mathbb{T}$ in trace class and any bounded operator $\mathbb{B} : \mathcal{H} \to \mathcal{H}$

- $\|\mathbb{T}\| \le \|\mathbb{T}\|_{\rm tr}$,
- $\|\mathbb{T}\|_{\rm tr} = \|\mathbb{T}^\star\|_{\rm tr}$,
- $\|\mathbb{B}\mathbb{T}\|_{\rm tr} \le \|\mathbb{B}\|_{\rm tr} \cdot \|\mathbb{T}\|_{\rm tr}$,
- $\|\mathbb{T}\mathbb{B}\|_{\rm tr} \le \|\mathbb{B}\|_{\rm tr} \cdot \|\mathbb{T}\|_{\rm tr}$.
- For any pair of trace class operators $\mathbb{T}$ and $\mathbb{S}$, $\mathbb{T}+\mathbb{S}$ is trace class, and $\|\mathbb{T}+\mathbb{S}\|_{\rm tr} \le \|\mathbb{T}\|_{\rm tr} + \|\mathbb{S}\|_{\rm tr}$.
- The trace class is a two-sided ideal in the algebra of all bounded linear operators on a complex Hilbert space.
- Trace class operators form a Banach space with respect to the trace norm.
- Every trace class operator is HS.
- The product of two HS operators $\mathbb{T}$ and $\mathbb{S}$ is in trace class, and $\|\mathbb{S}\mathbb{T}\|_{\rm tr} \le \|\mathbb{S}\|_{\rm HS} \cdot \|\mathbb{T}\|_{\rm HS}$.
- Every trace class operator can be written as the product of two HS operators.

Let $\{\mathbf{x}_n\}$ be any orthonormal basis of $\mathcal{H}$. For a trace class operator $\mathbb{T}$, the *trace* is defined as the limit of the series

$$\operatorname{tr}\mathbb{T} := \sum_n \langle \mathbb{T}\mathbf{x}_n, \mathbf{x}_n\rangle.$$

This series converges absolutely. For trace class operators $\mathbb{T}$

- $\operatorname{tr}\mathbb{T} = \sum_n \lambda_n(\mathbb{T})$, where $\lambda_n(\mathbb{T})$ are the eigenvalues of $\mathbb{T}$.
- If $\mathbb{T}$ is a trace class operator that has no eigenvalues except $\lambda = 0$. Then, $\operatorname{tr}\mathbb{T} = 0$.
- $|\operatorname{tr}\mathbb{T}| \le \|\mathbb{T}\|_{\rm tr}$.
- $\operatorname{tr}\mathbb{T}$ is a linear mapping of $\mathbb{T}$.
- $\operatorname{tr}\mathbb{T}^\star = (\operatorname{tr}\mathbb{T})^*$.
- For any bounded operator $\mathbb{B}$, $\operatorname{tr}(\mathbb{T}\mathbb{B}) = \operatorname{tr}(\mathbb{B}\mathbb{T})$.

Let $\mathbb{T}$ be a trace class operators, and let $\{\mathbb{T}_n\}$ be a sequence of *degenerate* operators, i.e., operators with finite range that converge to $\mathbb{T}$ in trace norm. Then, the determinant $\det(\mathbf{I}+\mathbf{T}_n)$ of the matrix representation of $\mathbb{I}+\mathbb{T}_n$, $\mathbf{I}+\mathbf{T}_n$, tends to a limit that is independent of the choice of the sequence $\{\mathbb{T}_n\}$. This limit is called the *determinant* of $\mathbb{I}+\mathbb{T}$:

$$\det(\mathbb{I}+\mathbb{T}) := \lim_{n\to\infty} \det(\mathbf{I}+\mathbf{T}_n).$$

The Spectral Theorem

Orthogonal Projection [2, 1]. A projection $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called an *orthogonal projection* if its range and null space are orthogonal: $\mathcal{R}(\mathbb{P}) \perp \mathcal{N}(\mathbb{P})$.

- A bounded linear operator $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is an orthogonal projection if $\mathbb{P}$ is self adjoint and *idempotent*, i.e., $\mathbb{P}^2 = \mathbb{P}$.
- An orthogonal projection is continuous (even if $\mathcal{H}$ is not complete).
- A continuous projection on a Hilbert space is orthogonal iff it is self-adjoint.
- $\mathbb{P} \succeq \mathbb{O}$;
- $\|\mathbb{P}\| \le 1$ with equality if $\mathbb{P}(\mathcal{H}) \ne \{\mathbf{0}\}$.
- $\mathcal{N}(\mathbb{P}) = \mathcal{R}(\mathbb{P})^\perp$ and $\mathcal{R}(\mathbb{P}) = \mathcal{N}(\mathbb{P})^\perp$.
- For any orthogonal projection $\mathbb{P}$ on a Hilbert space $\mathcal{H}$ and for any $\mathbf{h} \in \mathcal{H}$, $\langle \mathbb{P}\mathbf{h}, \mathbf{h}\rangle = \|\mathbb{P}\mathbf{h}\|^2$.
- Each $\mathbf{h} \in \mathcal{H}$ can be written uniquely as $\mathbf{r}+\mathbf{n}$, where $\mathbf{r} \in \mathcal{R}(\mathbb{P})$ and $\mathbf{n} \in \mathcal{N}(\mathbb{P})$; furthermore, $\|\mathbf{x}\|^2 = \|\mathbf{r}\|^2 + \|\mathbf{n}\|^2$.
- Let $\mathcal{M}$ be any closed subspace of a Hilbert space $\mathcal{H}$. Then there is exactly one orthogonal projection $\mathbb{P}$ with $\mathcal{R}(\mathbb{P}) = \mathcal{M}$. Let $\{\mathbf{e}_n\}$ be a countable orthonormal set in $\mathcal{H}$ such that $\mathcal{M} = \overline{\operatorname{span}\{\mathbf{e}_n\}}$; then, the mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ defined by
  $$\mathbb{P}\mathbf{h} := \sum_n \langle \mathbf{h}, \mathbf{e}_n\rangle \mathbf{e}_n$$
  for any $\mathbf{h} \in \mathcal{H}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.
- Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, let $\mathbf{h} \in \mathcal{H}$, and let $\mathbb{P}$ be the orthogonal projection on $\mathcal{H}$ with $\mathcal{R}(\mathbb{P}) = \mathcal{M}$. Then, $\|\mathbf{h}-\mathbb{P}\mathbf{h}\| = \inf_{\mathbf{m}\in\mathcal{M}} \|\mathbf{h}-\mathbf{m}\|$.

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be orthogonal projections on a Hilbert space $\mathcal{H}$, and let $\mathcal{Y}_1 := \mathbb{P}_1(\mathcal{H})$ and $\mathcal{Y}_2 := \mathbb{P}_2(\mathcal{H})$.

- The composite operator $\mathbb{P} := \mathbb{P}_1\mathbb{P}_2$ is a projection on $\mathcal{H}$ iff $\mathbb{P}_1$ and $\mathbb{P}_2$ commute. In this case, $\mathbb{P}$ projects $\mathcal{H}$ onto $\mathcal{Y} = \mathcal{Y}_1 \cap \mathcal{Y}_2$. Conversely, the projection onto $\overline{\operatorname{span}\{\mathcal{Y}_1, \mathcal{Y}_2\}}$ is $\mathbb{P}_1 + \mathbb{P}_2 - \mathbb{P}_1\mathbb{P}_2$.
- The sum $\mathbb{P} := \mathbb{P}_1 + \mathbb{P}_2$ is a projection operator iff $\mathcal{Y}_1 \perp \mathcal{Y}_2$. In this case, $\mathbb{P}$ projects $\mathcal{H}$ onto $\mathcal{Y}_1 \oplus \mathcal{Y}_2$.
- The difference $\mathbb{P} := \mathbb{P}_2 - \mathbb{P}_1$ is a projection on $\mathcal{H}$ iff $\mathcal{Y}_1 \subset \mathcal{Y}_2$. In this case, $\mathbb{P}$ projects $\mathcal{H}$ onto the orthogonal complement of $\mathcal{Y}_1$ in $\mathcal{Y}_2$.
- The following conditions are equivalent
  - $\mathbb{P}_2\mathbb{P}_1 = \mathbb{P}_1\mathbb{P}_2 = \mathbb{P}_1$,
  - $\mathcal{Y}_1 \subset \mathcal{Y}_2$,
  - $\mathcal{N}(\mathbb{P}_1) \supset \mathcal{N}(\mathcal{Y}_2)$,
  - $\|\mathbb{P}_1\mathbf{h}\| \le \|\mathbb{P}_2\mathbf{h}\|$ for all $\mathbf{h} \in \mathcal{H}$,
  - $\mathbb{P}_1 \preceq \mathbb{P}_2$.

Let $\{\mathbb{P}_n\}$ be a monotonically increasing sequence of projection operators $\mathbb{P}_n$ on a Hilbert space $\mathcal{H}$. Then,

- The sequence $\{\mathbb{P}_n\}$ is strongly operator convergent, say $\mathbb{P}_n\mathbf{h} \to \mathbb{P}\mathbf{h}$ for all $\mathbf{h} \in \mathcal{H}$, and the limit operator $\mathbb{P}$ is a projection on $\mathcal{H}$.
- The limit operator $\mathbb{P}$ projects $\mathcal{H}$ onto
  $$\mathbb{P}(\mathcal{H}) = \overline{\bigcup_{n=1}^{\infty} \mathbb{P}_n(\mathcal{H})}.$$
- The limit operator $\mathbb{P}$ has the null space
  $$\mathcal{N}(\mathbb{P}) = \bigcap_{n=1}^{\infty} \mathcal{N}(\mathbb{P}_n).$$

Spectral Family [1, 2]. A real *spectral family*, is a collection $\mathscr{E} := \{\mathbb{E}_\lambda : \lambda \in \mathbb{R}\}$ of projection operators $\mathbb{E}_\lambda$ on a Hilbert space $\mathcal{H}$ of any dimension that satisfy the following properties for any $\mathbf{h} \in \mathcal{H}$:

- $\mathbb{E}_\lambda \le \mathbb{E}_\mu$ for $(\lambda \le \mu)$; hence, $\mathbb{E}_\lambda\mathbb{E}_\mu = \mathbb{E}_\mu\mathbb{E}_\lambda = \mathbb{E}_\lambda$.
- $\lim_{\lambda\to-\infty} \mathbb{E}_\lambda\mathbf{h} = \mathbf{0}$.
- $\lim_{\lambda\to\infty} \mathbb{E}_\lambda\mathbf{h} = \mathbf{h}$.
- $\mathbb{E}_{\lambda+}\mathbf{h} := \lim_{\mu\downarrow\lambda} \mathbb{E}_\mu\mathbf{h} = \mathbb{E}_\lambda\mathbf{h}$.

Special cases:

- A countable *resolution of the identity* is a sequence $\{\mathbb{P}_n\}$ of orthogonal projection operators with $\mathbb{P}_n\mathbb{P}_m = \mathbb{O}$ for $n \ne m$ so that $\mathbb{I} = \sum_n \mathbb{P}_n$, where the sum is strongly operator convergent. The sequence $\{\mathbb{P}_n\}$ defines a spectral family $\mathscr{E}$ with
  $$\mathbb{E}_\lambda := \sum_{n\le\lambda} \mathbb{P}_n.$$
- A *spectral family on an interval* $[a,b] \in \mathbb{R}$ is a spectral family $\mathscr{E}$ that satisfies $\mathbb{E}_\lambda = \mathbb{O}$ for $\lambda < a$ and $\mathbb{E}_\lambda = \mathbb{I}$ for $\lambda \ge b$.

Compact Normal Operators [2]. Let $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be a compact normal operator on a Hilbert space $\mathcal{H}$.

- There is a countable resolution of the identity $\{\mathbb{P}_n\}$ and a sequence of complex numbers $\{\mu_n\}$ such that
  $$\mathbb{T} = \sum_n \mu_n\mathbb{P}_n,$$
  where convergence is uniform in the operator norm.
- There exists an orthonormal basis $\{\mathbf{e}_n\}$ of eigenvectors and a corresponding sequence of eigenvalues $\{\lambda_n\}$ such that, if $\mathbf{h} = \sum_n \langle \mathbf{h}, \mathbf{e}_n\rangle \mathbf{e}_n$ is the Fourier expansion for $\mathbf{h} \in \mathcal{H}$, then
  $$\mathbb{T}\mathbf{h} = \sum_n \lambda_n \langle \mathbf{h}, \mathbf{e}_n\rangle \mathbf{e}_n.$$
- A weighted sum of projections $\sum_n \lambda_n\mathbb{P}_n$, where $\{\mathbb{P}_n\}$ is a resolution of the identity, and $\{\lambda_n\}$ is a sequence of complex numbers, is compact if (i) for every nonzero λ_n, the range of $\mathbb{P}_n$ is finite dimensional, and (ii) for every real $\alpha > 0$, the number of λ_n with $|\lambda_n| \ge \alpha$ is finite.

Bounded Self-Adjoint Operators [1, §9]. Let $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint, linear operator on a Hilbert space $\mathcal{H}$, let $\mathbb{T}_\lambda := \mathbb{T} - \lambda\mathbb{I}$, and define the *positive part* of $\mathbb{T}_\lambda$ as $\mathbb{T}_\lambda^+ := ((\mathbb{T}_\lambda^2)^{1/2} + \mathbb{T}_\lambda)/2$. Furthermore, let $\mathcal{Y}_\lambda := \mathcal{N}(\mathbb{T}_\lambda^+)$ denote the null space of $\mathbb{T}_\lambda^+$.

- Let $\mathbb{E}_\lambda$ with $\lambda \in \mathbb{R}$ be the projection of $\mathcal{H}$ onto the null space $\mathcal{Y}_\lambda$ of $\mathbb{T}_\lambda^+$. Then, the collection $\mathscr{E}(\mathbb{T}) := \{\mathbb{E}_\lambda : \lambda \in \mathbb{R}\}$ is the unique spectral family associated with $\mathbb{T}$ on the interval $[m_\mathbb{T}, M_\mathbb{T}] \in \mathbb{R}$.
- For $\lambda < \mu$, the projection operator $\mathbb{E}_\mu - \mathbb{E}_\lambda$ satisfies $\lambda(\mathbb{E}_\mu - \mathbb{E}_\lambda) \preceq \mathbb{T}(\mathbb{E}_\mu - \mathbb{E}_\lambda) \preceq \mu(\mathbb{E}_\mu - \mathbb{E}_\lambda)$.
- The mapping $\lambda \to \mathbb{E}_\lambda$ has a discontinuity at λ_0, i.e., $\mathbb{E}_{\lambda_0} \ne \mathbb{E}_{\lambda_0^+}$, iff λ_0 is an eigenvalue of $\mathbb{T}$. In this case, the eigenspace that corresponds to the eigenvalue λ_0 is $\mathcal{N}(\mathbb{T}-\lambda_0\mathbb{I}) = (\mathbb{E}_{\lambda_0} - \mathbb{E}_{\lambda_0^+})(\mathcal{H})$.
- A real λ_0 belongs to the resolvent set $\mathbb{R}_\lambda(\mathbb{T})$ iff there is an $\epsilon > 0$ such that $\mathscr{E}(\mathbb{T})$ is constant on the interval $[\lambda_0-\epsilon, \lambda_0+\epsilon]$.
- A real λ_0 belongs to the continuous spectrum $\mathcal{S}_c(\mathbb{T})$ iff the mapping $\lambda \to \mathbb{E}_\lambda$ is continuous at λ_0 (thus $\mathbb{E}_{\lambda_0} = \mathbb{E}_{\lambda_0^+}$) and is not constant in any neighborhood of λ_0.

A bounded, self-adjoint, linear operator $\mathbb{T}$ on a complex Hilbert space $\mathcal{H}$ has the spectral representation

$$\mathbb{T} = \int_{m_\mathbb{T}-0}^{M_\mathbb{T}} \lambda d\mathbb{E}_\lambda = m_\mathbb{T}\mathbb{E}_{m_\mathbb{T}} + \int_{m_\mathbb{T}}^{M_\mathbb{T}} \lambda d\mathbb{E}_\lambda,$$

where $\mathscr{E} = \{\mathbb{E}_\lambda\}$ is the spectral family associated with $\mathbb{T}$, and the integral is to be understood in the sense of uniform operator convergence in the norm on $\mathcal{B}(\mathcal{H},\mathcal{H})$.

- For $\mathbf{x}, \mathbf{y} \in \mathcal{H}$,
  $$\langle \mathbb{T}\mathbf{x}, \mathbf{y}\rangle = \int_{m_\mathbb{T}-0}^{M_\mathbb{T}} \lambda dw(\lambda),$$
  where $w(\lambda) := \langle \mathbb{E}_\lambda\mathbf{x}, \mathbf{y}\rangle$, and the integral is of Riemann-Stieltjes type.
- Let $f(\lambda) : [m_\mathbb{T}, M_\mathbb{T}] \to \mathbb{R}$ be a continuous, real-valued function on $[m_\mathbb{T}, M_\mathbb{T}]$. Define $f(\mathbb{T})$ as the limit $p(\mathbb{T})$ of the polynomial $\mathbb{T}_n := p_n(\mathbb{T}) := \alpha_n\mathbb{T}^n + \alpha_{n-1}\mathbb{T}^{n-1} + \cdots + \alpha_0\mathbb{I}$ for $n \to \infty$, where $p_n(\lambda)$ is such that it converges uniformly to $f(\lambda)$ on $[m_\mathbb{T}, M_\mathbb{T}]$. Then, the operator $f(\mathbb{T})$ has the spectral representation
  $$f(\mathbb{T}) = \int_{m_\mathbb{T}-0}^{M_\mathbb{T}} f(\lambda) d\mathbb{E}_\lambda,$$
  and for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$,
  $$\langle f(\mathbb{T})\mathbf{x}, \mathbf{y}\rangle = \int_{m_\mathbb{T}-0}^{M_\mathbb{T}} f(\lambda) dw(\lambda).$$
  - The operator $f(\mathbb{T})$ is self adjoint.
  - If $f(\lambda) = f_1(\lambda)f_2(\lambda)$, then $f(\mathbb{T}) = f_1(\mathbb{T})f_2(\mathbb{T})$.
  - If $f(\lambda) \ge 0$ for all $\lambda \in [m_\mathbb{T}, M_\mathbb{T}]$, then $f(\mathbb{T}) \succeq \mathbb{O}$.
  - If $f_1(\lambda) \le f_2(\lambda)$ for all $\lambda \in [m_\mathbb{T}, M_\mathbb{T}]$, then $f_1(\mathbb{T}) \preceq f_2(\mathbb{T})$.
  - $\|f(\mathbb{T})\| \le \max_{\lambda\in[m_\mathbb{T},M_\mathbb{T}]} |f(\lambda)|$.
  - If a bounded linear operator commutes with $\mathbb{T}$, it also commutes with $f(\mathbb{T})$.

Unitary Operators [1, §10.5]. Let $\mathbb{U} : \mathcal{H} \to \mathcal{H}$ be a unitary operator on a complex Hilbert space $\mathcal{H}$.

- The spectrum $\mathcal{S}(\mathbb{U})$ is a closed subset of the unit circle. Consequently, $|\lambda| = 1$ for every $\lambda \in \mathcal{S}(\mathbb{U})$.
- There exists a spectral family $\mathscr{E} = \{\mathbb{E}_\lambda\}$ on $[-\pi, \pi]$ such that
  $$\mathbb{U} = \int_{-\pi}^{\pi} e^{i\lambda} d\mathbb{E}_\lambda.$$
- for every continuous function f on the unit circle,
  $$f(\mathbb{U}) = \int_{-\pi}^{\pi} f\left(e^{i\lambda}\right) d\mathbb{E}_\lambda,$$
  where the integral is to be understood in the sense of uniform operator convergence.
- For all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$,
  $$\langle f(\mathbb{U})\mathbf{x}, \mathbf{y}\rangle = \int_{-\pi}^{\pi} f\left(e^{i\lambda}\right) dw(\lambda),$$
  where $w(\lambda) := \langle \mathbb{E}_\lambda\mathbf{x}, \mathbf{y}\rangle$, and the integral is an ordinary Riemann-Stieltjes integral.

Linear Operator Equations

Fredholm Alternative [1, §8.7]. A bounded linear operator $\mathbb{S} : \mathcal{U} \to \mathcal{U}$ on a normed space is said to satisfy the *Fredholm alternative* if either one of the following conditions holds

- The nonhomogeneous equations
  $$\mathbb{S}\mathbf{x} = \mathbf{y}, \qquad \mathbb{S}^\times f = g$$
  have unique solutions $\mathbf{x}$ and f, respectively, for every given $\mathbf{y} \in \mathcal{U}$ and $g \in \mathcal{U}'$, and the corresponding homogeneous equations
  $$\mathbb{S}\mathbf{x} = \mathbf{0}, \qquad \mathbb{S}^\times f = o$$
  have only the trivial solutions $\mathbf{x} = \mathbf{0}$ and $f = 0$, respectively.
- The homogeneous equations
  $$\mathbb{S}\mathbf{x} = \mathbf{0}, \qquad \mathbb{S}^\times\mathbb{T} = 0$$
  have the same number of linearly independent solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ and $f_1, f_2, \ldots, f_N$, respectively, and the corresponding nonhomogeneous equations
  $$\mathbb{S}\mathbf{x} = \mathbf{y}, \qquad \mathbb{S}^\times f = g$$
  are not solvable for all $\mathbf{y}$ and f, respectively. They have a solution iff $\mathbf{y}$ and g are such that $f_n(\mathbf{y}) = 0$ and $g(\mathbf{x}_n) = 0$ for all $n = 1, 2, \ldots, N$.

For a compact linear operator $\mathbb{T}$ on a normed space $\mathcal{U}$, the operator $\mathbb{T}_\lambda := \mathbb{T} - \lambda\mathbb{I}$, for $\lambda \ne 0$, satisfies the Fredholm alternative.

Linear Operator Equations [1, §8.5]. Let $\mathbb{T} : \mathcal{U} \to \mathcal{U}$ be a compact linear operator on a normed space $\mathcal{U}$ and $\mathbb{T}^\times : \mathcal{U}' \to \mathcal{U}'$ its adjoint operator. For $\mathbf{x}, \mathbf{y} \in \mathcal{U}$, $f, g \in \mathcal{U}'$, and $\lambda \ne 0$, consider the set of linear operator equations

$$\mathbb{T}\mathbf{x} - \lambda\mathbf{x} = \mathbf{y} \qquad (\mathbf{y} \in \mathcal{X} \text{ given}) \qquad \text{(OE1)}$$
$$\mathbb{T}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \qquad \text{(OE2)}$$
$$\mathbb{T}^\times f - \lambda f = g \qquad (g \in \mathcal{X}' \text{ given}) \qquad \text{(OE3)}$$
$$\mathbb{T}^\times f - \lambda f = 0. \qquad \text{(OE4)}$$

Then,

- Equation (OE1) has a solution $\mathbf{x}$ iff $f(\mathbf{y}) = 0$ for all solutions f of (OE4). Hence, if $f = 0$ is the only solution of (OE4), then (OE1) is solvable for every $\mathbf{y}$.
- Equation (OE3) has a solution f iff $g(\mathbf{x}) = 0$ for all solutions $\mathbf{x}$ of (OE2). Hence, if $\mathbf{x} = \mathbf{0}$ is the only solution of (OE2), then (OE3) is solvable for every g.
- Equation (OE1) has a solution $\mathbf{x}$ for every $\mathbf{y} \in \mathcal{U}$ iff $\mathbf{x} = \mathbf{0}$ is the only solution of (OE2).
- Equation (OE3) has a solution f for every $g \in \mathcal{U}'$ iff $f = 0$ is the only solution of (OE4).
- Equations (OE2) and (OE4) have the same number of linearly independent solutions.

References

[1] E. Kreyszig, *Introduction to Functional Analysis with Applications*. New York, NY, U.S.A.: Wiley, 1989.

[2] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science*. New York, NY, U.S.A.: Springer, 1982.

[3] M. Pourahmadi, *Foundations of Time Series Analysis and Prediction Theory*. New York, NY, U.S.A.: Wiley, 2001.

[4] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York, NY, U.S.A: McGraw-Hill, 1987.

[5] ——, *Functional Analysis*, 2nd ed. New York, NY, U.S.A.: McGraw-Hill, 1991.

[6] N. Dunford and J. T. Schwarz, *Linear Operators*, New York, NY, U.S.A., 1958, vol. 1.

[7] R. L. Allen and D. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*. Piscataway, NJ, U.S.A.: IEEE Press, 2004.

[8] P. D. Lax, *Functional Analysis*. New York, NY, U.S.A.: Wiley, 2002.

[9] N. Dunford and J. T. Schwarz, *Linear Operators*. New York, NY, U.S.A.: Wiley, 1963, vol. 2.