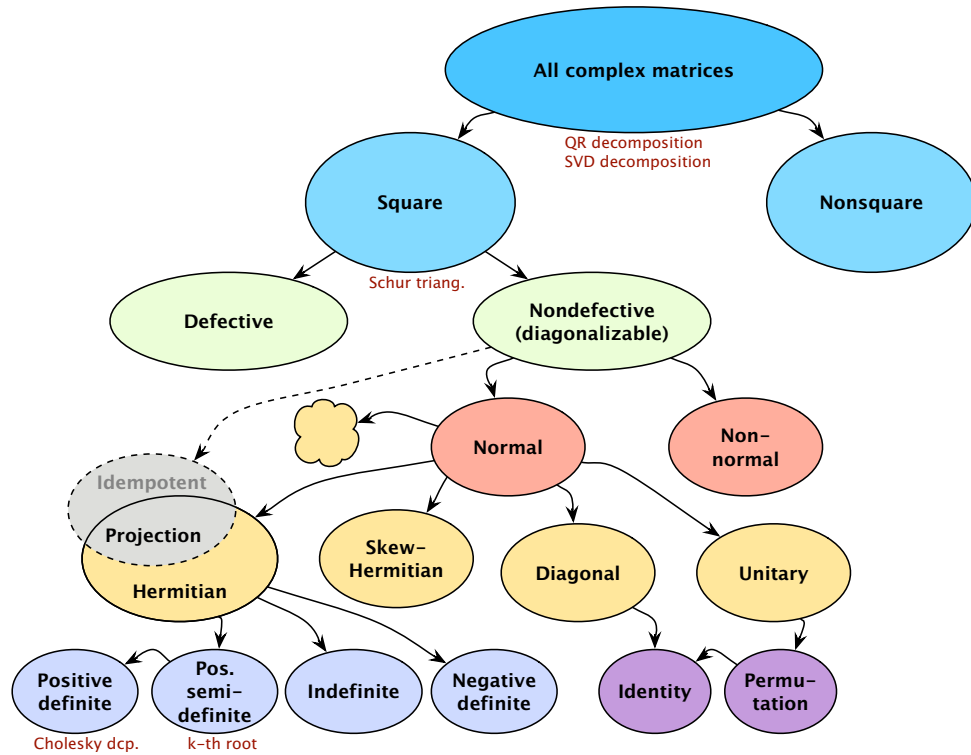
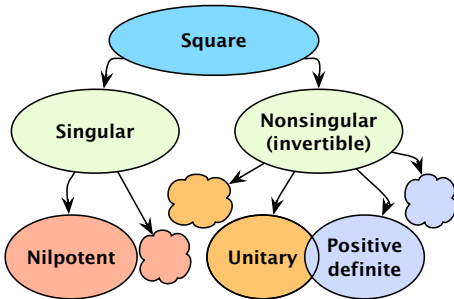


Matrix Taxonomy



Nonsingular Matrices (0.5).



A matrix $\mathbf{A} \in \mathcal{M}_n$ is nonsingular iff

- \mathbf{A}^{-1} exists
- $\det \mathbf{A} \neq 0$
- $\text{rnk } \mathbf{A} = n$
- $\dim \mathcal{R}(\mathbf{A}) = n$
- rows (cols) are linearly independent
- $0 \notin \sigma(\mathbf{A})$

Nilpotent Matrices (p39). A matrix $\mathbf{A} \in \mathcal{M}_n$ is nilpotent to index k if $\mathbf{A}^k = \mathbf{0}$ but

Hermitian Matrices (4.1.1). A matrix $\mathbf{A} \in \mathcal{M}_n$ is Hermitian if $\mathbf{A} = \mathbf{A}^H$.

- A Hermitian matrix is parametrized by n^2 free real variables.
- (4.1.1.1) For any $\mathbf{B} \in \mathcal{M}_n$, $\mathbf{B} + \mathbf{B}^H$, $\mathbf{B}^H \mathbf{B}$, and $\mathbf{B} \mathbf{B}^H$ are Hermitian.
- (4.1.1.2,3) Hermitian matrices are closed under addition, multiplication by a scalar, raising to an integer power, and (if nonsingular) inversion.
- (4.1.2) Any matrix \mathbf{A} has a unique decomposition $\mathbf{A} = \mathbf{B} + j\mathbf{C}$ where \mathbf{B} and \mathbf{C} are Hermitian: $\mathbf{B} = (\mathbf{A} + \mathbf{A}^H)/2$ and $\mathbf{C} = (\mathbf{A} - \mathbf{A}^H)/(2j)$
- (4.1.3c) The eigenvalues of a Hermitian matrix are all real.
- (4.1.4a) \mathbf{A} is Hermitian iff $\mathbf{x}^H \mathbf{A} \mathbf{x}$ is real for all $\mathbf{x} \in \mathbb{C}^n$.
- (4.1.4c) \mathbf{A} is Hermitian iff $\mathbf{S}^H \mathbf{A} \mathbf{S}$ is Hermitian for all $\mathbf{S} \in \mathcal{M}_n$ (or: iff $\mathbf{x}^H \mathbf{A} \mathbf{y} =$

Matrix Properties

Eigenvalues and -vectors (1.1.2). If $\lambda \in \mathbb{C}$ and $\mathbf{x} \neq \mathbf{0}$ satisfy the equation $\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$, they are considered eigenvalue and eigenvector, respectively, of the matrix $\mathbf{A} \in \mathcal{M}_n$. The *eigenvalue spectrum* $\sigma(\mathbf{A})$ of \mathbf{A} is the set of all eigenvalues.

- (1.2.4) The eigenvalue spectrum $\sigma(\mathbf{A})$ coincides with the roots of the *characteristic polynomial* $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$.
- (1.2.6) Every $\mathbf{A} \in \mathcal{M}_n$ has exactly n eigenvalues, counting multiplicities (c.m.).
- (2.4.2, Cayley-Hamilton) $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$. If \mathbf{A} is nonsingular, then there is a polynomial $q(t)$ of degree at most $n - 1$, such that $\mathbf{A}^{-1} = q(\mathbf{A})$.
- (1.4.1) The eigenvalues of \mathbf{A}^T are the same as those of \mathbf{A} , c.m. The eigenvalues of \mathbf{A}^H are the complex conjugates of the eigenvalues of \mathbf{A} , c.m.
- (1.4.2) The set of eigenvectors associated with a particular $\lambda \in \sigma(\mathbf{A})$ is a subspace of \mathbb{C}^n , called *eigenspace* of \mathbf{A} corresponding to λ .
- (1.4.3) The dimension of the eigenspace of $\mathbf{A} \in \mathcal{M}_n$ corresponding to the eigenvalue λ is its *geometric multiplicity*.
- (1.4.3) The multiplicity of λ as a zero of $p_{\mathbf{A}}(t)$ is the *algebraic multiplicity* of λ . The algebraic multiplicity is greater or equal than the geometric multiplicity.
- (1.3.8) Eigenvectors corresponding to different eigenvalues are linearly independent.
- (1.3.20) Be $\mathbf{A} \in \mathcal{M}_{m,n}$ and $\mathbf{B} \in \mathcal{M}_{n,m}$. Then $\mathbf{B} \mathbf{A}$ has the same eigenvalues as $\mathbf{A} \mathbf{B}$, c.m., together with an additional $n - m$ eigenvalues equal to 0.
- (p37) For nonsingular $\mathbf{A} \in \mathcal{M}_n$, if $\lambda \in \sigma(\mathbf{A})$, then $\lambda^{-1} \in \sigma(\mathbf{A}^{-1})$, corresponding to the same eigenvector.
- (p43) If $\lambda \in \sigma(\mathbf{A})$, then $\lambda^k \in \sigma(\mathbf{A}^k)$.
- If $\lambda \in \sigma(\mathbf{A})$, then $(1 + \lambda) \in \sigma(\mathbf{I} + \mathbf{A})$.
- (4.3.1, Weyl) For Hermitian $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n$, with their eigenvalues arranged in increasing order, for $k = 1, \dots, n$:

$$\lambda_k(\mathbf{A}) + \lambda_1(\mathbf{B}) \leq \lambda_k(\mathbf{A} + \mathbf{B}) \leq \lambda_k(\mathbf{A}) + \lambda_n(\mathbf{B}).$$
- (4.5.9, Ostrowski) For $\mathbf{A} \in \mathcal{M}_n$ Hermitian

Submatrices (0.7). With index sets $\alpha \subseteq \{1, \dots, m\}$, $\beta \subseteq \{1, \dots, n\}$ and $\mathbf{A} \in \mathcal{M}_{m,n}$, $\mathbf{A}(\alpha, \beta)$ denotes the submatrix that is indexed by rows α and columns β .

- For $\mathbf{A} \in \mathcal{M}_n$, if $\alpha = \beta$, $\mathbf{A}(\alpha, \beta) = \mathbf{A}(\alpha)$ is a *principal submatrix* of \mathbf{A} .
- The determinant of a square submatrix of $\mathbf{A} \in \mathcal{M}_{m,n}$ is called a *minor*.
- The determinant of a principal submatrix of $\mathbf{A} \in \mathcal{M}_n$ is a *principal minor*.

Adjoint Matrix (0.8.2). The adjoint of $\mathbf{A} \in \mathcal{M}_n$ is the matrix $\mathbf{B} \in \mathcal{M}_n$ of *cofactors* defined by

$$b_{ij} = (-1)^{i+j} \det \mathbf{A}_{ji}.$$

- $(\text{adj } \mathbf{A}) \mathbf{A} = \mathbf{A} (\text{adj } \mathbf{A}) = (\det \mathbf{A}) \mathbf{I}$.
- If \mathbf{A} invertible, $\mathbf{A}^{-1} = (\det^{-1} \mathbf{A}) \text{adj } \mathbf{A}$.

Range and Null Space (0.2.3). The range of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is

$\mathcal{R}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{C}^m : \mathbf{y} = \mathbf{A} \mathbf{x}, \mathbf{x} \in \mathbb{C}^n\}$.
The null space of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is

- $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{A} \mathbf{x} = \mathbf{0}\}$.
- (0.2.3) $\dim \mathcal{R}(\mathbf{A}) + \dim \mathcal{N}(\mathbf{A}) = n$.
- (0.4.4g) $\dim \mathcal{R}(\mathbf{A}) = \text{rnk } \mathbf{A}$.
- (0.6.6) $\mathcal{R}(\mathbf{A}) = \mathcal{N}^\perp(\mathbf{A}^H)$.

Rank (0.4). The rank of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is the largest number of rows (columns) of \mathbf{A} that constitute a linearly independent set.

- (0.4.4) $\text{rnk } \mathbf{A} = \dim \mathcal{R}(\mathbf{A}) = n - \dim \mathcal{N}(\mathbf{A})$.
- (0.4.6a) $\mathbf{A}, \mathbf{A}^H, \mathbf{A}^T, \mathbf{A}^*$ have the same rank.
- (0.4.6b) rank is unchanged by left/right multiplication with a nonsingular matrix.
- (0.4.6c) If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$, then $\text{rnk } \mathbf{A} = \text{rnk } \mathbf{B}$ iff there are nonsingular $\mathbf{X} \in \mathcal{M}_m$ and $\mathbf{Y} \in \mathcal{M}_n$ s.t. $\mathbf{B} = \mathbf{X} \mathbf{A} \mathbf{Y}$.
- (0.4.6d) $\text{rnk } \mathbf{A}^H \mathbf{A} = \text{rnk } \mathbf{A}$.
- (0.4.5a) $\text{rnk } \mathbf{A} \leq \min(m, n)$.
- (0.4.5c) If $\mathbf{A} \in \mathcal{M}_{m,k}$ and $\mathbf{B} \in \mathcal{M}_{k,n}$

$$\text{rnk } \mathbf{A} + \text{rnk } \mathbf{B} - k \leq \text{rnk } \mathbf{A} \mathbf{B}$$

$$\text{rnk } \mathbf{A} \mathbf{B} \leq \min(\text{rnk } \mathbf{A}, \text{rnk } \mathbf{B}).$$
- (0.4.5d) If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$, $\text{rnk}(\mathbf{A} + \mathbf{B}) \leq \text{rnk } \mathbf{A} + \text{rnk } \mathbf{B}$.
- (p95) For $\mathbf{A} \in \mathcal{M}_n$,

$\mathbf{A}^{k-1} \neq \mathbf{0}^*$

- (p37) $\lambda_i = 0, i = 1, \dots, n..$

Idempotent Matrices (p37). A matrix $\mathbf{P} \in \mathcal{M}_n$ is idempotent if $\mathbf{P}^2 = \mathbf{P}$.

- $\text{rk } \mathbf{P} = \text{tr } \mathbf{P}$.
- (p37) $\lambda_i \in \{0, 1\}, i = 1, \dots, n$. The geometric multiplicity of the eigenvalue 1 is $\text{rk } \mathbf{P}$.
- $\mathbf{P}^H, \mathbf{I} - \mathbf{P}$, and $\mathbf{I} - \mathbf{P}^H$ are all idempotent.
- $\mathbf{P}(\mathbf{I} - \mathbf{P}) = (\mathbf{I} - \mathbf{P})\mathbf{P} = \mathbf{0}$.
- $\mathbf{P}\mathbf{x} = \mathbf{x}$ iff \mathbf{x} lies in the range of \mathbf{P} .
- $\mathcal{N}(\mathbf{P}) = \mathcal{R}(\mathbf{I} - \mathbf{P})$.
- (TODO) \mathbf{P} is its own generalized inverse, \mathbf{P}^\dagger .

Nondefective (Diagonalizable) Matrices (1.3.6). A matrix $\mathbf{A} \in \mathcal{M}_n$ is diagonalizable if it is similar to a diagonal matrix, i.e., $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$.

- (1.3.7) The columns of \mathbf{S} are eigenvectors of \mathbf{A} , the diagonal entries of $\mathbf{\Lambda}$ corresponding eigenvalues.
- (1.4.4) \mathbf{A} is diagonalizable iff it is nondefective, i.e., the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue.
- (1.3.9) If \mathbf{A} has n distinct eigenvalues, it is diagonalizable.
- (p46) $\text{rk } \mathbf{A} = \#$ nonzero eigenvalues.

Normal Matrices (2.5). A matrix $\mathbf{A} \in \mathcal{M}_n$ is normal if $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$.

- (2.5.4) \mathbf{A} is normal iff it is unitarily similar to a diagonal matrix.
- (2.5.4) \mathbf{A} is normal iff it has an orthonormal set of n eigenvectors.
- Normal matrices are closed under unitary equivalence, raising to an integer power, and (if nonsingular) inversion.
- The singular values of a normal matrix are the absolute values of the eigenvalues.
- The eigenvalues of \mathbf{A}^H are the conjugates of the eigenvalues of \mathbf{A} and have the same eigenvectors.
- A normal matrix is Hermitian iff its eigenvalues are all real.
- A normal matrix is skew-Hermitian iff its eigenvalues all have zero real parts.
- A normal matrix is unitary iff its eigenvalues all have an absolute value of 1.
- If \mathbf{A} and \mathbf{B} are normal and $\mathbf{AB} = \mathbf{BA}$ then \mathbf{AB} is normal.
- (2.5.4) $\|\mathbf{A}\|_{\mathbb{F}}^2 = \sum_{i=1}^n |\lambda_i|^2$.

$\mathbf{x}^H\mathbf{A}^H\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$).

- (4.1.5, Spectral Theorem) \mathbf{A} is Hermitian iff there is a unitary matrix $\mathbf{U} \in \mathcal{M}_n$ and a real diagonal matrix $\mathbf{\Lambda} \in \mathcal{M}_n$ s.t. $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$.
- If \mathbf{A} and \mathbf{B} are Hermitian then so are $\mathbf{AB} + \mathbf{BA}$ and $j(\mathbf{AB} - \mathbf{BA})$.
- (4.2.2, Rayleigh-Ritz) $\min(\lambda) \leq \mathbf{x}^H\mathbf{A}\mathbf{x} \leq \max(\lambda), \mathbf{x}^H\mathbf{x} = 1$.

Positive Definite Matrices (4.1.1). A Hermitian matrix $\mathbf{A} \in \mathcal{M}_n$ is *positive definite* (pd) if $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$. If $\mathbf{x}^H\mathbf{A}\mathbf{x} \geq 0$, \mathbf{A} is said to be *positive semidefinite* (psd).

- (7.1.2) Any principal submatrix of a pd matrix is pd.
- (7.1.3) Any nonnegative linear combination of psd matrices is psd.
- (7.1.4,5) For p(s)d $\mathbf{A} \in \mathcal{M}_n, \lambda_i$ (for $i = 1, \dots, n$), $\det \mathbf{A}$, $\text{tr } \mathbf{A}$, and all principal minors are positive (nonnegative).
- (7.2.1) $\mathbf{A} \in \mathcal{M}_n$ is p(s)d iff all eigenvalues are positive (nonnegative).
- (7.2.8) $\mathbf{A} \in \mathcal{M}_n$ is pd iff there is a nonsingular $\mathbf{C} \in \mathcal{M}_n$ s.t. $\mathbf{A} = \mathbf{C}^H\mathbf{C}$. Any solution \mathbf{C} can be written as $\mathbf{C} = \mathbf{V}\mathbf{A}^{1/2}$ with $\mathbf{V} \in \mathcal{M}_n$ unitary.
- (p409) Any psd rank- m matrix $\mathbf{A} \in \mathcal{M}_n$ may be written as $\mathbf{A} = \mathbf{C}^H\mathbf{C}$ with some $\mathbf{C} \in \mathcal{M}_{m,n}$.
- (7.1.6) For pd $\mathbf{A} \in \mathcal{M}_n$ and $\mathbf{C} \in \mathcal{M}_{n,m}$, $\mathbf{C}^H\mathbf{A}\mathbf{C}$ is psd, and $\text{rk } \mathbf{C}^H\mathbf{A}\mathbf{C} = \text{rk } \mathbf{C}$.

Skew-Hermitian Matrices. A matrix $\mathbf{K} \in \mathcal{M}_n$ is skew-Hermitian if $\mathbf{K} = -\mathbf{K}^H$.

- (p175) \mathbf{S} is Hermitian iff $j\mathbf{S}$ is skew-Hermitian.
- \mathbf{K} is skew-Hermitian iff $\mathbf{x}^H\mathbf{K}\mathbf{y} = -\mathbf{x}^H\mathbf{K}^H\mathbf{y}$ for all \mathbf{x} and \mathbf{y} .
- Skew-Hermitian matrices are closed under addition, multiplication by a scalar, raising to an odd power, and (if nonsingular) inversion.
- (p175) \mathbf{K}^2 is Hermitian.
- (p175) The eigenvalues of a skew-Hermitian matrix are either 0 or pure imaginary.

Triangular Matrices (0.9.3). A matrix $\mathbf{T} \in \mathcal{M}_n$ is upper (lower) triangular if $t_{ij} = 0$ for $j < i$ ($j > i$). Strictness if true for $j \leq i$ ($j \geq i$).

- Upper/lower triangular matrices are closed under addition, multiplication,

and $\mathbf{S} \in \mathcal{M}_n$ nonsingular, with all eigenvalues arranged in increasing order, for all $k = 1, \dots, n$ there is a positive real number θ_k in the range

$$\lambda_1(\mathbf{S}\mathbf{S}^H) \leq \theta_k \leq \lambda_n(\mathbf{S}\mathbf{S}^H) \quad \text{s.t.}$$

$$\lambda_k(\mathbf{S}\mathbf{A}\mathbf{S}^H) = \theta_k \lambda_k(\mathbf{A}).$$

Inverse Matrix. For any $\mathbf{A} \in \mathcal{M}_{m,n}$, having a SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$, there is a *Moore-Penrose generalized inverse* or *pseudoinverse* $\mathbf{A}^\dagger = \mathbf{V}\mathbf{\Sigma}^\dagger\mathbf{U}^H$, where $\mathbf{\Sigma}^\dagger$ is the transpose of $\mathbf{\Sigma}$ in which the positive singular values are replaced by their reciprocals.

- (p421) $\mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\dagger$ are Hermitian.
- (p421) $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$ and $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$.
- (0.5) For any nonsingular $\mathbf{A} \in \mathcal{M}_n$, there is a unique $\mathbf{A}^\dagger = \mathbf{A}^{-1}$ s.t. $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.
- Any full-rank skinny $\mathbf{A} \in \mathcal{M}_{m,n}, m \geq n$, has a left inverse $\mathbf{B} \in \mathcal{M}_{n,m}$ s.t. $\mathbf{B}\mathbf{A} = \mathbf{I}_n$. The left inverse with the smallest norm is the pseudoinverse $\mathbf{B} = \mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.
- Any full-rank fat $\mathbf{A} \in \mathcal{M}_{m,n}, m \leq n$, has a right inverse $\mathbf{B} \in \mathcal{M}_{n,m}$ s.t. $\mathbf{A}\mathbf{B} = \mathbf{I}_m$. The right inverse with the smallest norm is the pseudoinverse $\mathbf{B} = \mathbf{A}^\dagger = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}$.
- $(\mathbf{A}^H)^\dagger = (\mathbf{A}^\dagger)^H$, and $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$ (if \mathbf{A} nonsingular).
- (0.7.4, Matrix Inversion Lemma) Be $\mathbf{A} \in \mathcal{M}_m$ and $\mathbf{R} \in \mathcal{M}_n$ nonsingular, and $\mathbf{X} \in \mathcal{M}_{m,n}$ and $\mathbf{Y} \in \mathcal{M}_{n,m}$. Then

$$(\mathbf{A} + \mathbf{X}\mathbf{R}\mathbf{Y})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}(\mathbf{R}^{-1} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X})^{-1}\mathbf{Y}\mathbf{A}^{-1}$$

$$(c\mathbf{I}_m + \mathbf{X}\mathbf{Y})^{-1} = \frac{1}{c} \left(\mathbf{I}_m - \mathbf{X}(c\mathbf{I}_n + \mathbf{Y}\mathbf{X})^{-1}\mathbf{Y} \right)$$

$$(c\mathbf{I}_m + \mathbf{X}\mathbf{X}^H)^{-1} = \frac{1}{c} \left(\mathbf{I}_m - \mathbf{X}(c\mathbf{I}_n + \mathbf{X}^H\mathbf{X})^{-1}\mathbf{X}^H \right).$$

- For $\mathbf{X} \in \mathcal{M}_{m,n}$ and $\mathbf{Y} \in \mathcal{M}_{n,m}$, $\mathbf{Y}(c\mathbf{I} + \mathbf{X}\mathbf{Y})^{-1} = (c\mathbf{I} + \mathbf{Y}\mathbf{X})^{-1}\mathbf{Y}$.
- For nonsingular $\mathbf{A}, \mathbf{B} \in \mathcal{M}_m$, $\mathbf{A}^{-1} = \mathbf{B}^{-1} + \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1}$.
- If $\mathbf{A} \in \mathcal{M}_m$ and the inverses exist, any pair of $\mathbf{A}, (c\mathbf{I} - \mathbf{A})^{-1}$, and $(c\mathbf{I} + \mathbf{A})^{-1}$ commutes. Further,

$$\mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I}$$

$\text{rk } \mathbf{A} \geq \#$ nonzero eigenvalues.

(p46) Equality for diagonalizable \mathbf{A} .

Trace (0.4). For $\mathbf{A} \in \mathcal{M}_n, \text{tr } \mathbf{A} = \sum_{i=1}^n a_{ii}$.

- (1.2.12) $\text{tr } \mathbf{A} = \sum_{i=1}^n \lambda_i(\mathbf{A})$.
- $\text{tr}(\alpha\mathbf{A}) = \alpha \text{tr}(\mathbf{A})$.
- $\text{tr}(\mathbf{A}^T) = \text{tr}(\mathbf{A})$.
- $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$.
- $\text{tr}(\mathbf{A}\mathbf{B}) = \text{tr}(\mathbf{B}\mathbf{A})$.
- $\text{tr}(\mathbf{a}\mathbf{b}^T) = \mathbf{a}^T\mathbf{b}$.
- $\text{tr}(\mathbf{a}\mathbf{b}^H) = \mathbf{a}^H\mathbf{b}$.
- $\text{tr}(\mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D}) = \text{tr}(\mathbf{B}\mathbf{C}\mathbf{D}\mathbf{A}) = \text{tr}(\mathbf{C}\mathbf{D}\mathbf{A}\mathbf{B}) = \text{tr}(\mathbf{D}\mathbf{A}\mathbf{B}\mathbf{C})$.
- $\text{tr}(\mathbf{A} \otimes \mathbf{B}) = \text{tr}(\mathbf{A}) \text{tr}(\mathbf{B})$.

Determinant (0.3). For $\mathbf{A} \in \mathcal{M}_n$,

$$\det \mathbf{A} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij} = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij},$$

where $\mathbf{A}_{ij} \in \mathcal{M}_{n-1}$ is the submatrix obtained by deleting row i and column j .

- (1.2.12) $\det \mathbf{A} = \prod_{i=1}^n \lambda_i(\mathbf{A})$.
- (0.3.1) $\det \mathbf{A}^T = \det \mathbf{A}$.
- (0.3.1) $\det \mathbf{A}^H = \overline{\det \mathbf{A}}$.
- $\det c\mathbf{A} = c^n \det \mathbf{A}$.
- Interchanging any pair of columns of \mathbf{A} multiplies $\det \mathbf{A}$ by -1 (likewise rows).
- Multiplying any column of \mathbf{A} by c multiplies $\det \mathbf{A}$ by c (likewise rows).
- Adding any multiple of one column onto another column leaves $\det \mathbf{A}$ unaltered (likewise rows).
- For $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n, \det \mathbf{A}\mathbf{B} = \det \mathbf{A} \det \mathbf{B}$.
- For nonsingular $\mathbf{A} \in \mathcal{M}_n, \det(\mathbf{A} + \mathbf{x}\mathbf{y}^H) = \det(\mathbf{A})(1 + \mathbf{y}^H\mathbf{A}^{-1}\mathbf{x})$.
- If $\mathbf{A} \in \mathcal{M}_n$ and $\mathbf{D} \in \mathcal{M}_k$,

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{Q} = \det(\mathbf{A}) \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) = \det(\mathbf{D}) \det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}).$$

The quantity $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is called the *Schur complement* of \mathbf{A} in \mathbf{Q} .

For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{M}_n$

$$\mathbf{A}\mathbf{C} = \mathbf{C}\mathbf{A} : \det \mathbf{Q} = \det(\mathbf{A}\mathbf{D} - \mathbf{C}\mathbf{B})$$

$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} : \det \mathbf{Q} = \det(\mathbf{D}\mathbf{A} - \mathbf{C}\mathbf{B})$$

$$\mathbf{D}\mathbf{B} = \mathbf{B}\mathbf{D} : \det \mathbf{Q} = \det(\mathbf{D}\mathbf{A} - \mathbf{C}\mathbf{B})$$

$$\mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} : \det \mathbf{Q} = \det(\mathbf{A}\mathbf{D} - \mathbf{C}\mathbf{B})$$

- (7.8.1, Hadamard's inequality) If \mathbf{A} is positive semidefinite, then $\det \mathbf{A} \leq \prod_{i=1}^n a_{ii}$.

- (p111) The direct sum $\mathbf{A}_1 \oplus \dots \oplus \mathbf{A}_k$ is normal iff each \mathbf{A}_j is normal.

Projection Matrices. A matrix $\mathbf{P} \in \mathcal{M}_n$ is a projection matrix if it is Hermitian and idempotent, i.e., $\mathbf{P}^H = \mathbf{P}^2 = \mathbf{P}$.

- \mathbf{P} is positive semidefinite.
- $\mathbf{I} - \mathbf{P}$ is a projection matrix.
- With any subspace $\mathcal{S} \subseteq \mathbb{C}^n$, any vector $\mathbf{x} \in \mathbb{C}^n$ can be decomposed as $\mathbf{x} = \mathbf{P}_{\mathcal{S}} \mathbf{x} + \mathbf{P}_{\mathcal{S}}^{\perp} \mathbf{x}$ with $\mathbf{P}_{\mathcal{S}}^{\perp} = \mathbf{I}_n - \mathbf{P}_{\mathcal{S}}$ where $\mathbf{P}_{\mathcal{S}}$ is the unique projection matrix on \mathcal{S} . Also, $\mathcal{R}(\mathbf{P}_{\mathcal{S}}) = \mathcal{S}$.
- Be $\mathcal{S} = \mathcal{R}(\mathbf{A})$. Then the projection matrix on \mathcal{S} is $\mathbf{P}_{\mathcal{S}} = \mathbf{A}\mathbf{A}^{\dagger}$.
- If $\mathbf{B} \in \mathcal{M}_{n,k}$ is a basis for the subspace \mathcal{S} , then $\mathbf{P}_{\mathcal{S}} = \mathbf{B}(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H$.
- If \mathbf{U} is a unitary basis for the subspace \mathcal{S} , then $\mathbf{P}_{\mathcal{S}} = \mathbf{U}\mathbf{U}^H$.
- If \mathbf{P} and \mathbf{Q} are projection matrices, then the following are equivalent:
 - $\mathbf{P} - \mathbf{Q}$ is a projection matrix
 - $\mathbf{P} - \mathbf{Q}$ is positive semidefinite
 - $\mathbf{P}\mathbf{Q} = \mathbf{Q}$
 - $\mathbf{Q}\mathbf{P} = \mathbf{Q}$

Nonderogatory Matrices (1.4.4). A matrix $\mathbf{A} \in \mathcal{M}_n$ is nonderogatory if every eigenvalue has geometric multiplicity 1.

raising to an integer power, and (if non-singular) inversion.

- $\lambda_i = t_{ii}$, $i = 1, \dots, n$.
- $\det \mathbf{T} = \prod_{i=1}^n t_{ii}$.
- $\text{rk } \mathbf{T} \geq \#$ of nonzero t_{ii} .
- (0.9.4, p62) For a block-triangular

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

the eigenvalues are those of $\mathbf{A} \in \mathcal{M}_n$ together with those of $\mathbf{C} \in \mathcal{M}_m$, c.m. Thus, $\det \mathbf{T} = \det \mathbf{A} \det \mathbf{C}$. Further,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix}.$$

Unitary Matrices (2.1). A complex matrix $\mathbf{U} \in \mathcal{M}_n$ is unitary if $\mathbf{U}^H\mathbf{U} = \mathbf{I}$.

- Unitary matrices are closed under multiplication, raising to an integer power, and inversion.
- \mathbf{U} is unitary iff \mathbf{U}^H is unitary.
- \mathbf{U} is unitary iff $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$ for all \mathbf{x} .
- $|\lambda_i| = 1$, $i = 1, \dots, n$.
- $|\det \mathbf{U}| = 1$.
- A square matrix is unitary iff its columns form an orthonormal basis.
- The space of unitary matrices in \mathcal{M}_n is parametrized by n^2 free real variables and called *unitary group*.
- \mathbf{U} is unitary iff $\mathbf{U} = e^{j\mathbf{K}}$ or $j\mathbf{K} = \ln(\mathbf{U})$ for some Hermitian \mathbf{K} (no 1:1 mapping).
- (Cayley transform) \mathbf{U} is unitary iff $\mathbf{U} = (\mathbf{I} + j\mathbf{K})^{-1}(\mathbf{I} - j\mathbf{K})$ for some Hermitian \mathbf{K} .

Permutation Matrices (0.9.5). $\mathbf{A} \in \mathcal{M}_n$ is a permutation matrix if its columns are a permutation of the columns of \mathbf{I} .

Matrix Relations

Similarity (1.3). A matrix $\mathbf{A} \in \mathcal{M}_n$ is similar to a matrix $\mathbf{B} \in \mathcal{M}_n$ if there exists a nonsingular matrix $\mathbf{S} \in \mathcal{M}_n$ s.t. $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

- (p43) Similarity invariants are: trace, determinant, rank, characteristic polynomial, eigenvalues.

Unitary Equivalence (2.2). A matrix $\mathbf{A} \in \mathcal{M}_n$ is unitarily equivalent (or unitarily similar) to a matrix $\mathbf{B} \in \mathcal{M}_n$ if there exists a unitary matrix $\mathbf{U} \in \mathcal{M}_n$ s.t. $\mathbf{B} = \mathbf{U}^H\mathbf{A}\mathbf{U}$.

- (2.2.2) Additional *unitary* similarity invariants are: Frobenius norm (and thus $\text{tr } \mathbf{A}^H\mathbf{A}$).

$$\mathbf{A}(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} - (\mathbf{I} + \mathbf{A})^{-1}.$$

Decompositions

Minimum-Rank Factorization (0.4.6e). Every rank- k $\mathbf{A} \in \mathcal{M}_{m,n}$ may be written as $\mathbf{A} = \mathbf{X}\mathbf{B}\mathbf{Y}$ with $\mathbf{X} \in \mathcal{M}_{m,k}$, $\mathbf{Y} \in \mathcal{M}_{k,n}$, and $\mathbf{B} \in \mathcal{M}_k$. In particular, a rank-1 matrix may be written as $\mathbf{A} = \mathbf{xy}^H$.

Singular Value Decomposition (7.3.5). Every rank- k matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ may be written as

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H,$$

where $\mathbf{U} \in \mathcal{M}_m$ and $\mathbf{V} \in \mathcal{M}_n$ are unitary. The matrix $\mathbf{\Sigma} \in \mathcal{M}_{m,n}$ contains the k nonzero entries $\sigma_{11}, \dots, \sigma_{kk}$ in nonincreasing order and zeros elsewhere.

- The real *singular values* $\sigma_i = \sigma_{ii}$ are the nonnegative square roots of the eigenvalues of $\mathbf{A}\mathbf{A}^H$ and therefore unique.
- The columns of \mathbf{U} are eigenvectors of $\mathbf{A}\mathbf{A}^H$, the columns of \mathbf{V} are eigenvectors of $\mathbf{A}^H\mathbf{A}$.
- (p422) The first k columns of \mathbf{U} form an orthonormal basis for the range of \mathbf{A} , the last $n-k$ columns of \mathbf{V} form an orthonormal basis for the null space of \mathbf{A} .
- (p418) The singular values are invariant under conjugation, transposition, and left or right multiplication with a unitary matrix.
- $\text{rk } \mathbf{A} = \#$ nonzero singular values.

Schur Triangularization (2.3). Every $\mathbf{A} \in \mathcal{M}_n$ is unitarily similar to an upper triangular matrix, i.e., $\mathbf{U}^H\mathbf{A}\mathbf{U} = \mathbf{T}$, with $t_{ii} = \lambda_i$, $i = 1, \dots, n$. Neither \mathbf{U} nor \mathbf{T} are unique.

QR Factorization (2.6). Every $\mathbf{A} \in \mathcal{M}_{n,m}$ with $n \geq m$ can be written as $\mathbf{A} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q} \in \mathcal{M}_{n,m}$ has orthonormal columns and $\mathbf{R} \in \mathcal{M}_m$ is upper triangular.

- If \mathbf{A} is nonsingular, then \mathbf{R} may be chosen s.t. all r_{ii} are positive, in which event both \mathbf{Q} and \mathbf{R} are unique.

Cholesky Decomposition (7.2.9). A matrix $\mathbf{A} \in \mathcal{M}_n$ is positive definite iff there is a nonsingular lower triangular $\mathbf{L} \in \mathcal{M}_n$ with positive diagonal entries s.t. $\mathbf{A} = \mathbf{L}\mathbf{L}^H$ (or an upper triangular \mathbf{R} with $\mathbf{A} = \mathbf{R}^H\mathbf{R}$).

k th Root (7.2.6). For any positive (semi)definite matrix $\mathbf{A} \in \mathcal{M}_n$, there is a unique positive (semi)definite Hermitian k th ($k > 0$) root $\mathbf{B} \in \mathcal{M}_n$ s.t. $\mathbf{B}^k = \mathbf{A}$. This k th root is denoted by $\mathbf{B} = \mathbf{A}^{1/k}$.

- $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$.
- $\text{rk } \mathbf{B} = \text{rk } \mathbf{A}$.
- \mathbf{B} is positive definite iff \mathbf{A} is.
- $\mathbf{B} = p(\mathbf{A})$ for some polynomial $p(\cdot)$.
- (p54) Every diagonalizable matrix has a square root.

Matrix Operators

Kronecker Product (II 4). For $\mathbf{A} \in \mathcal{M}_{m,n}$ and $\mathbf{B} \in \mathcal{M}_{p,q}$,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \dots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

- (II 4.2) $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$
- $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$
- $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$.
- (II 4.2.10) Mixed products: $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{A}\mathbf{C} \otimes \mathbf{B}\mathbf{D}$.
- Inversion: If \mathbf{A} and \mathbf{B} nonsingular, then $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$.

- (II 4.3) $\text{vec}(\mathbf{A}\mathbf{B}\mathbf{C}) = (\mathbf{C}^T \otimes \mathbf{A}) \text{vec } \mathbf{B}$
- $\text{vec}(\mathbf{A}\mathbf{B}) = (\mathbf{B}^T \otimes \mathbf{I}) \text{vec } \mathbf{A}$
- $= (\mathbf{I} \otimes \mathbf{A}) \text{vec } \mathbf{B}$.

- (II p. 252) $\text{vec}^T(\mathbf{Y})(\mathbf{A} \otimes \mathbf{B}) \text{vec}(\mathbf{X}) = \text{tr}(\mathbf{A}^T\mathbf{Y}^T\mathbf{B}\mathbf{X})$

Hadamard Product (II 5). For $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$

$$\mathbf{A} \circ \mathbf{B} = [a_{ij}b_{ij}] \in \mathcal{M}_{m,n}.$$

- (II 5.1.7) $\text{rk } \mathbf{A} \circ \mathbf{B} \leq \text{rk } \mathbf{A} \text{rk } \mathbf{B}$.
- (II p311) $\det \mathbf{A} \circ \mathbf{B} \geq \det \mathbf{A} \det \mathbf{B}$.