ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Chair for Mathematical Information Science

Prof. Dr. H. Bölcskei
Sternwartstrasse 7
CH-8092 Zürich

Mathematics of Information

Spring semester 2022

Problem Set 1

Problem 1 Unconditional convergence. ☕

Let $\mathcal{H} = l_2(\mathbb{N})$ and define

$$x_k = (0, \dots, 0, 1/k, 0, \dots, 0, \dots), \quad k \in \mathbb{N},$$

where the only non-zero entry $1/k$ of the sequence x_k is at position $k \in \mathbb{N}$. Does the sum

$$\sum_{k=1}^{\infty} x_k$$

converge unconditionally?

Problem 2 Cauchy-Schwarz inequality. ☕

a) Prove that the elements x and y of a complex Hilbert space $\mathcal{H}$ satisfy $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Hint: First expand and simplify the quantity $\left\| \|y\|^2 x - \langle x, y\rangle y \right\|^2$.

b) Prove that if the elements x and y of a complex Hilbert space $\mathcal{H}$ satisfy $|\langle x, y\rangle| = \|x\|\|y\|$ and $y \neq 0$, then $x = cy$ for some $c \in \mathbb{C}$.

Hint: Assume $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = 1$. Then $x - y$ and x are orthogonal, while $x = x - y + y$. Therefore, $\|x\|^2 = \|x - y\|^2 + \|y\|^2$.

Problem 3 A norm inequality. ☕

Prove that the inequalities

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

hold for every $x \in \mathbb{R}^n$.

Problem 4 The spaces L^1 and L^2. ☕

Throughout this problem, f will denote a function from $\mathbb{R}$ to $\mathbb{R}$.

a) We first consider

$$f \colon x \mapsto \mathbb{1}_{(0,1]}(x)\frac{1}{\sqrt{x}}.$$

Prove that f is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$.

b) Now suppose $f \in L^1(\mathbb{R})$ is an arbitrary bounded function, that is, there exists a real number $A < \infty$ such that $|f(x)| \leq A$, for all $x \in \mathbb{R}$. Prove that then $f \in L^2(\mathbb{R})$ and

$$\|f\|_{L^2(\mathbb{R})} \leq \sqrt{A \, \|f\|_{L^1(\mathbb{R})}}.$$

c) Now suppose that $f \in L^2(\mathbb{R})$, and further assume that there exists finite real numbers $a < b$ such that f vanishes outside of the interval $[a, b]$. Prove that $f \in L^1(\mathbb{R})$.

Problem 5 Projection on closed subspaces. ☕☕

Let $\mathcal{H}$ be a Hilbert space, $x \in \mathcal{H}$, and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. We know that there exists a $y \in \mathcal{S}$ such that

$$\|x - y\| = \min_{z \in \mathcal{S}} \|x - z\|. \tag{1}$$

a) Use the parallelogram law

$$\|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2), \ \forall a, b \in \mathcal{H}.$$

to show that y is the unique element in $\mathcal{S}$ fulfilling (1).

b) Show that y is the unique element in $\mathcal{S}$ such that $(x - y) \in \mathcal{S}^{\perp}$.

Problem 6 A surjective linear isometry is unitary. ☕☕

Let $\mathcal{H}$ be a real Hilbert space and $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ a surjective linear isometry. By applying the polarization formula

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right), \quad \forall x, y \in \mathcal{H},$$

show that $\langle \mathbb{T}x, \mathbb{T}y \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. Deduce that $\mathbb{T}$ is unitary. Where did you use the fact that $\mathbb{T}$ is surjective?

Problem 7 Continuty and FT (Exam 2019, Problem 1). ☕☕☕

a) Show that the function $f_1(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$, is an element of $L^2(\mathbb{R})$ and compute $\|f_1\|_{L^2(\mathbb{R})}$. Show that f_1 is not an element of $L^1(\mathbb{R})$.

b) Present an explicit example of a function $f_2 \in L^1(\mathbb{R})$ that is not an element of $L^2(\mathbb{R})$.

Hint: Such a function f_2 will necessarily be unbounded.

In the remainder of the problem you may use, without proof, the following fact: If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then its Fourier transform $\hat{g} : \mathbb{R} \to \mathbb{C}$ is a continuous function.

c) Let f be an element of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and additionally assume that the function $H_f : \mathbb{R} \to \mathbb{C}$ given by

$$H_f(\omega) = \omega \, \hat{f}(\omega), \quad \omega \in \mathbb{R},$$

is an element of $L^2(\mathbb{R})$. Define the function $G_f : \mathbb{R} \to \mathbb{C}$ by

$$G_f(\omega) = (1 + |\omega|)|\hat{f}(\omega)|, \quad \omega \in \mathbb{R}.$$

i) Show that G_f is an element of $L^2(\mathbb{R})$.

ii) Show that $\hat{f}$ is an element of $L^1(\mathbb{R})$ by proving the following bound:

$$\|\hat{f}\|_{L^1(\mathbb{R})} \leq \sqrt{2}\,\|G_f\|_{L^2(\mathbb{R})}.$$

Hint: Apply the Cauchy-Schwarz inequality in the Hilbert space $L^2(\mathbb{R})$ to the functions $\omega \mapsto \frac{1}{1+|\omega|}$ and G_f.

iii) Show that f is a continuous function.

Problem 8 Parallelogram law. ☕☕☕

This exercise considers the so-called parallelogram law. Recall that a norm $\|\cdot\|$ on a linear space E over $\mathbb{R}$ is said to satisfy the parallelogram law if

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \ \forall x, y \in E.$$

As we will see in this exercise, this is a powerful tool to check whether a norm on a linear space E is induced by an inner product.

a) Let H be a Hilbert space together with its inner product $\langle\cdot,\cdot\rangle_H$ and the induced norm

$$\|x\|_H = \sqrt{\langle x, x\rangle_H}, \ \forall x \in H.$$

i) Prove that the norm $\|\cdot\|_H$ satisfies the parallelogram law.

ii) Prove the parallelogram law for n elements $x_1, \ldots, x_n \in H$ for the norm $\|\cdot\|_H$, that is,

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_n)\,\in\,\{-1,+1\}^n} \Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|_H^2 = 2^n \sum_{i=1}^n \|x_i\|_H^2.$$

Hint: first develop the left-hand-side using the expression

$$\Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|_H^2 = \Big\langle \sum_{i=1}^n \varepsilon_i x_i, \sum_{j=1}^n \varepsilon_j x_j \Big\rangle_H$$

and prove that, for $i \neq j$, it holds that

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_n)\,\in\,\{-1,+1\}^n} \varepsilon_i \varepsilon_j = 0.$$

b) We are now interested in the converse. More precisely, we wish to prove that, if a normed linear space $(E, \|\cdot\|_E)$ over $\mathbb{R}$ satisfies the parallelogram law, then E can be equipped with an inner product $\langle\cdot,\cdot\rangle_E$ which satisfies

$$\langle x, x\rangle_E = \|x\|_E^2, \ \forall x \in E.$$

Concretely, we consider the candidate $\langle\cdot,\cdot\rangle$ defined as

$$\langle x, y\rangle \coloneqq \frac{1}{4}(\|x+y\|_E^2 - \|x-y\|_E^2), \ \forall x, y \in E.$$

i) Prove that $\langle \cdot, \cdot \rangle$ satisfies the following conditions
   - $\langle x, x\rangle = \|x\|_E^2, \ \forall x \in E$;
   - $\langle x, x\rangle = 0$ implies $x = 0$;
   - $\langle x, y\rangle = \langle y, x\rangle, \ \forall x, y \in E$.

ii) Prove that $\langle \cdot, \cdot \rangle$ is additive, that is,

$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle, \ \forall x_1, x_2, y \in E.$$

Hint: first prove, using the parallelogram law, that

$$\begin{aligned}\|x_1 + x_2 + 2y\|_E^2 &= 2\|x_1 + y\|_E^2 + 2\|x_2 + y\|_E^2 - \|x_1 - x_2\|_E^2 \\ &= 2\|x_1 + x_2 + y\|_E^2 + 2\|y\|_E^2 - \|x_1 + x_2\|_E^2,\end{aligned}$$

and

$$\begin{aligned}\|x_1 + x_2 - 2y\|_E^2 &= 2\|x_1 - y\|_E^2 + 2\|x_2 - y\|_E^2 - \|x_1 - x_2\|_E^2 \\ &= 2\|x_1 + x_2 - y\|_E^2 + 2\|y\|_E^2 - \|x_1 + x_2\|_E^2.\end{aligned}$$

iii) Prove that

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle, \ \forall x, y \in E, \ \forall \lambda \in \mathbb{R}.$$

Hint: first prove the relation for λ in $\mathbb{N}$, then in $\mathbb{Z}$, then in $\mathbb{Q}$ and then argue by density to prove the desired result.

c) Use the last subproblem to show that the Banach space $\ell^2(\mathbb{Z})$ is actually a Hilbert space. What about the space $\ell^1(\mathbb{Z})$?

Recall: $\ell^p(\mathbb{Z})$, $p \in [1, \infty)$, is the space

$$\ell^p(\mathbb{Z}) \triangleq \left\{ \{u_k\}_{k\in\mathbb{Z}} \colon \sum_{k=-\infty}^{+\infty} |u_k|^p < \infty \right\}$$

equipped with the norm

$$\|u\|_{\ell^p(\mathbb{Z})} \triangleq \left(\sum_{k=-\infty}^{+\infty} |u_k|^p \right)^{1/p}.$$

Problem 9 Short-time Fourier Transform (Exam 2018, Problem 1). ☕☕☕

Let $f, g \in L^2(\mathbb{R})$. We define the short-time Fourier transform $V_g f : \mathbb{R}^2 \to \mathbb{C}$ of f with respect to window g by

$$(V_g f)(x, \omega) = \int_{\mathbb{R}} f(t)\overline{g(t - x)} e^{-2\pi i \omega t} \mathrm{d}t.$$

Consider the following two transformations:

- The *asymmetric coordinate transform* $\mathcal{T}_a$ is defined for a function $F : \mathbb{R}^2 \to \mathbb{C}$ by

$$\mathcal{T}_a F(x, y) = F(y, y - x).$$

- The *partial Fourier transform* $\mathcal{F}_2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the unique bounded linear operator on $L^2(\mathbb{R}^2)$ with the following property:
  Whenever $F \in L^2(\mathbb{R}^2)$ is such that $F(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$, meaning that
  $$\int_{\mathbb{R}} |F(x,y)| \, \mathrm{d}y < \infty \quad \text{for all } x \in \mathbb{R},$$
  $\mathcal{F}_2 F$ is given by the formula
  $$(\mathcal{F}_2 F)(x,\omega) = \int_{\mathbb{R}} F(x,t) e^{-2\pi i \omega t} \mathrm{d}t, \quad \text{for all } (x,\omega) \in \mathbb{R}^2. \tag{2}$$
  You may use, without proof, the fact that $\mathcal{F}_2$ is a unitary operator.

a) Show that $\mathcal{T}_a$ is a unitary operator on $L^2(\mathbb{R}^2)$.

   Hint: First show that $\mathcal{T}_a$ maps $L^2(\mathbb{R}^2)$ functions to $L^2(\mathbb{R}^2)$ functions, and then compute the adjoint $\mathcal{T}_a^*$ explicitly.

b) Show carefully that
   $$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g}) \quad \text{for all } f, g \in L^2(\mathbb{R}), \tag{3}$$
   where for two functions $h_1, h_2 : \mathbb{R} \to \mathbb{C}$ we write $h_1 \otimes h_2$ to denote the function $(h_1 \otimes h_2)(x,y) = h_1(x) h_2(y)$.

   Please note that if you want to use (2) *to compute $\mathcal{F}_2 F$ for a function $F \in L^2(\mathbb{R}^2)$, you first have to verify that F satisfies the additional assumption that $F(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$.*

c) Using (3) deduce that $V_g f \in L^2(\mathbb{R}^2)$ for all $f, g \in L^2(\mathbb{R})$, and that
   $$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}, \quad \text{for all } f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}).$$

Problem 10 The Fourier transform on $L^2(\mathbb{R})$. ☢

This problem is more involved. You can find a complete and detailed analysis in the 'Fourier transform' notes on the webpage.

The Fourier transform on $\mathbb{R}$ is an operator mapping a function $f : \mathbb{R} \to \mathbb{C}$ to another function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by the integral
$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i \omega t} \mathrm{d}t, \quad \forall \omega \in \mathbb{R}.$$

a) Is the Fourier transform of $f : \mathbb{R} \to \mathbb{C}$ always well defined for
   i) $f \in L^1(\mathbb{R})$?
   ii) $f \in L^2(\mathbb{R})$?

b) Let $\varphi(t) = e^{-\pi t^2}$ be the normalized Gaussian. What is the Fourier transform of φ?

c) For $x, \xi \in \mathbb{R}$ we define the following linear operators on $L^2(\mathbb{R})$:
   $$(T_x f)(t) = f(t - x) \qquad \forall t \in \mathbb{R},$$
   $$(M_\xi f)(t) = e^{2\pi i \xi t} f(t) \qquad \forall t \in \mathbb{R},$$

where T_x is called the *translation* by x, and M_ξ is called the *modulation* by ξ. What are the Fourier transforms of $M_\xi T_x f$ and of $T_x M_\xi f$?

d) Let $\mathcal{X} = \operatorname{span}\{M_\xi T_x \varphi : (x, \xi) \in \mathbb{R}^2\}$. Show that we have the following properties

i) $\mathcal{X}$ is a subspace of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

ii) The Fourier transform $f \mapsto \hat{f}$ is a bijection from $\mathcal{X}$ to itself,

iii) $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, for all $f \in \mathcal{X}$,

iv) $\mathcal{X}$ is dense in $L^2(\mathbb{R})$,

and that they imply that the Fourier transform on $\mathcal{X}$ extends to a unitary operator on $L^2(\mathbb{R})$.