ETH

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Mathematics of Information

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Problem Set 10

Problem 1 Metric entropy of set of sequences ☕☕

a) Fix $n \in \mathbb{N}$, $\epsilon < 2^{-n}$, and consider the interval $I_n := [-2^{-n}, 2^{-n}] \in \mathbb{R}$ equipped with the metric $\rho_1(x, x') = |x - x'|$. Let $K := \log_2(1/\epsilon)$ and $L := \lceil 2^{K-n} \rceil$.

   i) Construct an ϵ-covering $A_n(\epsilon)$ of the interval I_n as follows. Divide I_n into L sub-intervals of equal length and show that the corresponding interval centers constitute an ϵ-covering of I_n.

   ii) Construct a 2ϵ-packing $P_n(\epsilon)$ of the interval I_n such that $|A_n| = |P_n|$.
   Hint. *Divide I_n into $L - 1$ sub-intervals and keep in mind that $|A_n| = |P_n|$.*

   iii) Compute $N(\epsilon; I_n, \rho_1)$.

b) Let $\mathcal{C} = \{f : \mathbb{N} \to \mathbb{R};\ f(n) \in [-2^{-n}, 2^{-n}],\ \forall n \in \mathbb{N}\}$ be a set of sequences in the space of bounded sequences equipped with the metric $\rho_2(f, g) = \sup_{n\in\mathbb{N}} |f(n) - g(n)|$.

   i) For $\epsilon \leq 1/2$, construct an ϵ-covering of $\mathcal{C}$.

   ii) Show that for every $\epsilon \leq 1/2$,

$$N(\epsilon; \mathcal{C}, \rho_2) \leq \left(\frac{1}{\epsilon}\right)^{\frac{1}{2}\log_2(1/\epsilon)+C},$$

   for some $C > 0$, which does not depend on ϵ.

Hints.

- *Use the result from subproblem (b.i) for $\epsilon < 2^{-n}$. For $\epsilon \geq 2^{-n}$, you can use, without proof, that $A_n(\epsilon) = \{0\}$ constitutes an ϵ-covering with $N(\epsilon; I_n, \rho_1) = 1$.*
- *You may use, without proof, that $\lceil 2^{K-n} \rceil \leq 2^{\lceil K \rceil - n}$, for $n \leq \lceil K \rceil - 1$.*

Problem 2 Minkowski dimension ☕☕

Given a compact set $K \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, we define the Minkowski dimension of K with respect to the norm $\|\cdot\|$ as

$$\dim_{\|\cdot\|}(K) := \lim_{\varepsilon \to 0^+} \frac{\log_2 \mathcal{N}(\varepsilon; K, \|\cdot\|)}{\log_2(1/\varepsilon)}, \tag{1}$$

where $\mathcal{N}(\varepsilon; K, \|\cdot\|)$ denotes the ε-covering number of K with respect to the norm $\|\cdot\|$, and $\varepsilon \in (0, 1)$. We will only consider compact sets K for which the limit (1) exists.

a) i) Fix $x \in \mathbb{R}^n$ and show that

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the usual 1- and 2-norm, respectively.

ii) Show that the result in (a)(i) implies the following inequalities between the corresponding ε-covering numbers

$$\mathcal{N}(\varepsilon; K, \|\cdot\|_2) \le \mathcal{N}(\varepsilon; K, \|\cdot\|_1) \le \mathcal{N}(\varepsilon/\sqrt{n}; K, \|\cdot\|_2). \tag{2}$$

iii) Deduce from (2) that

$$\dim_{\|\cdot\|_1}(K) = \dim_{\|\cdot\|_2}(K).$$

iv) Show that the Minkowski dimension of K is independent of the choice of the norm on $\mathbb{R}^n$, i.e., given two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathbb{R}^n$, we have

$$\dim_{\|\cdot\|}(K) = \dim_{\|\cdot\|'}(K).$$

We will denote this common quantity by $\dim(K)$, without subscript, hereafter and refer to it simply as "the Minkowski dimension".

Hint: Use the equivalence of norms in finite dimensions.

b) i) Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, prove that the Minkowski dimension of the ball $B_{\|\cdot\|}(0, R)$ (with respect to the norm $\|\cdot\|$) centered at the origin and of radius $R > 0$ satisfies $\dim(B_{\|\cdot\|}(0, R)) = n$, where the unsubscripted quantity $\dim(\cdot)$ is as defined in subproblem (a)(iv).

Hint: First prove the result in the case $R = 1$ using the relation between metric entropy and the volume ratio (cf. Handout). Then argue, for general $R > 0$, that $\dim(B_{\|\cdot\|}(0, R)) = \dim(B_{\|\cdot\|}(0, 1))$, using, without proof, the scaling relation $\mathcal{N}(\varepsilon; B_{\|\cdot\|}(0, 1), \|\cdot\|) = \mathcal{N}(R\varepsilon; B_{\|\cdot\|}(0, R), \|\cdot\|)$, for all $\varepsilon > 0$.

ii) Show that the Minkowski dimension of every compact set $K \subset \mathbb{R}^n$ is bounded according to $\dim(K) \le n$.

Hint: Use the result from subproblem (b)(i).

iii) Provide an example of a compact set $K \subset \mathbb{R}^n$ with $\dim(K) < n$.

Problem 3 Metric entropy scaling of Lipschitz functions

Consider the following class of functions

$$\mathcal{F}_L[a, b] := \{f : [a, b] \to \mathbb{R} \mid f(a) = 0, \text{ and } |f(x) - f(x')| \le L|x - x'|, \ \forall x, x' \in [a, b]\}.$$

Further let

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$$

be the sup norm on $\mathcal{F}_L[a, b]$ and $\rho_\infty(f, g) := \|f - g\|_\infty$ the metric it induces. (Note that we restrict to Lipschitz functions that satisfy $f(a) = 0$ in order to ensure compactness.)
The goal of this session is to prove that the metric entropy scaling of this function class is given by

$$\log_2 N(\epsilon; \mathcal{F}_L[a, b], \rho_\infty) \sim \frac{(b - a)L}{\epsilon}, \quad \text{for } \epsilon \to 0. \tag{3}$$

In order to do so, solve the following problems.

a) *Prove Lemma 1.*

Definition 1 *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. A mapping $i\colon X \to Y$ is called an isometry if $\rho_Y(i(x), i(x')) = \rho_X(x, x')$ for all $x, x' \in X$. A bijective isometry is called isometric isomorphism.*

Lemma 1 *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces and assume that there exists an isometric isomorphism $i\colon X \to Y$. Then*

$$N(\epsilon; X, \rho_X) = N(\epsilon; Y, \rho_Y).$$

Hint. Show that $\{x_1, \dots, x_n\}$ is a covering of X if and only if $\{i(x_1), \dots, i(x_n)\}$ is a covering of Y.

b) Let $\Delta := (b-a)L$ and consider the mapping

$$\begin{aligned} i\colon \mathcal{F}_L[a,b] &\to \mathcal{F}_1[0,\Delta] \\ f &\mapsto i(f), \end{aligned}$$

where $[i(f)](x) = f(\frac{x}{L} + a)$.
Show that i is an isometric isomorphism.

c) *Combine the above to establish that $N(\epsilon; \mathcal{F}_L[a,b], \rho_\infty) = N(\epsilon; \mathcal{F}_1[0,\Delta], \rho_\infty)$.*

d) Let $\epsilon > 0$ with $\Delta/\epsilon \in \mathbb{N}$. Let $n := \frac{\Delta}{\epsilon}$ and $t_k := k\epsilon$ for $k \in \{0, \dots, n\}$. We can partition the interval $[0, \Delta]$ into the n segments

$$I_k := [t_{k-1}, t_k], \quad k \in \{1, \dots, n\}.$$

Consider now the following set of continuous piecewise linear functions, which are zero on the segment I_1 and on each of the segments I_k, $k \geq 2$, have a slope of either 1 or -1. (See Figure 1 for an illustration).

$$\begin{aligned} S_\epsilon := \{\varphi \in C([0,\Delta])\colon \varphi\big|_{I_1} = 0 \text{ and} \\ \exists (b_2, \dots, b_n) \in \{-1, 1\}^{n-1}\colon \forall k \in \{2, \dots, n\}\colon (\varphi\big|_{I_k})'(x) = b_k\}. \end{aligned}$$

Show that S_ϵ constitutes an ϵ-covering of the set $\mathcal{F}_1[0,\Delta]$, and determine $|S_\epsilon|$.

e) Let $\epsilon > 0$ (not necessarily with $\Delta/\epsilon \in \mathbb{N}$).
Show that $N(\epsilon; \mathcal{F}_1[0,\Delta], \rho_\infty) \leq 2^{\frac{\Delta}{\epsilon}}$.
Hint. Show and use that, in general, $N(\epsilon; X, \rho) \leq N(\epsilon'; X, \rho)$ if $\epsilon \geq \epsilon'$.

f) *Use the approach from d) but with $n = \lceil \frac{\Delta}{\epsilon} \rceil - 1$ to construct a 2ϵ-packing of $\mathcal{F}_1[0,\Delta]$, and show that $M(2\epsilon; \mathcal{F}_1[0,\Delta], \rho_\infty) \geq 2^{\frac{\Delta}{\epsilon}-1}$.*

g) *Combine the above to establish* (3)*.*
Hint. Note that in general we have $M(2\epsilon; X, \rho_X) \leq N(\epsilon; X, \rho_X)$.

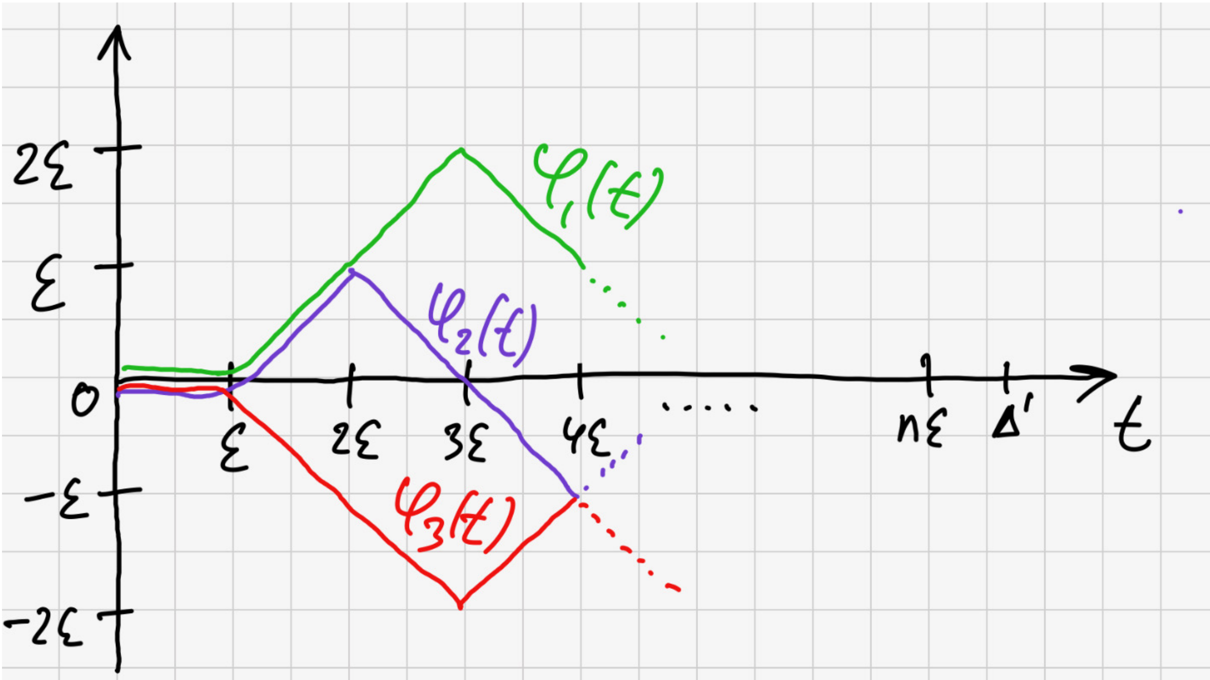

Figure 1: Some examples of functions in S_ϵ.