ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Chair for Mathematical Information Science

Prof. Dr. H. Bölcskei
Sternwartstrasse 7
CH-8092 Zürich

Mathematics of Information

Spring semester 2022

Problem Set 11

Problem 1 Integrated formula for the expectation. ☕

Prove that, for a nonnegative random variable X with finite expectation $\mathbb{E}[X] < \infty$, the following holds

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq s] \, ds.$$

Problem 2 Median vs Mean. ☕☕

Let X be a real valued random variable with finite variance, i.e., $\mathbb{V}[X] < \infty$. A median M of X is a real number such that

$$\mathbb{P}[X \geq M] \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}[X \leq M] \geq \frac{1}{2}.$$

The goal of this problem is to bound the difference between the mean and the median of a random variable.

a) Prove that the mean minimizes the L^2 loss, i.e.,

$$\mathbb{E}\left[|X - \mathbb{E}[X]|^2\right] = \inf_{c \in \mathbb{R}} \mathbb{E}\left[|X - c|^2\right].$$

b) Prove that the median minimizes the L^1 loss, i.e.,

$$\mathbb{E}\left[|X - M|\right] = \inf_{c \in \mathbb{R}} \mathbb{E}\left[|X - c|\right].$$

c) Using the result of subproblem b), prove that

$$|M - \mathbb{E}[X]| \leq \sqrt{\mathbb{V}[X]}.$$

Problem 3 Properties of sub-Gaussian random variables. ☕☕

In this problem, X denotes a sub-Gaussian random variable parameter $\sigma > 0$. Recall from the lecture that a random variable X with finite expectation $\mathbb{E}[X]$ is said to be sub-Gaussian with parameter $\sigma > 0$ if

$$\mathbb{E}\left[e^{\lambda(X - \mathbb{E}[X])}\right] \leq e^{\frac{\sigma^2 \lambda^2}{2}}, \quad \forall \lambda \in \mathbb{R}.$$

a) Prove that $\mathbb{V}[X] \leq \sigma^2$.

b) Now assume that $\mathbb{E}[X] = 0$. Prove that X satisfies $\mathbb{E}[e^{\frac{sX^2}{2\sigma^2}}] \leq \frac{1}{\sqrt{1-s}}$, for all $s \in [0, 1)$.

Hint: Multiply both sides in the definition of sub-Gaussian random variables by $e^{-\frac{\lambda^2\sigma^2}{2s}}$*, and then integrate w.r.t.* λ.

Problem 4 Bounded random variables are sub-Gaussian. ☕☕☕

Consider a random variable X with mean $\mu := \mathbb{E}[X]$, and such that, for some scalars $a < b$, $X \in [a, b]$ almost surely. We will show in this problem that X is sub-Gaussian.

a) Defining the function $\psi\colon \lambda \mapsto \log \mathbb{E}[e^{\lambda X}]$, show that $\psi(0) = 0$ and $\psi'(0) = \mu$.

b) Show that

$$\psi''(\lambda) = \frac{\mathbb{E}\left[X^2 e^{\lambda X}\right]}{\mathbb{E}\left[e^{\lambda X}\right]} - \frac{\mathbb{E}\left[X e^{\lambda X}\right]^2}{\mathbb{E}\left[e^{\lambda X}\right]^2}$$

and use this fact to obtain an upper bound on $\sup_{\lambda \in \mathbb{R}} |\psi''(\lambda)|$.

c) Using the previous questions, show that X is sub-Gaussian with parameter at most $\sigma := \frac{b-a}{2}$.

Problem 5 Convergence of random variables. ☕☕☕

Consider a sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that $\{X_n\}_{n \in \mathbb{N}}$ is said to converge *in probability* to a random variable X if

$$\mathbb{P}\left[|X_n - X| > \varepsilon\right] \xrightarrow[n \to \infty]{} 0, \quad \forall \varepsilon > 0.$$

The sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to converge *almost surely* (a.s.) to a random variable X if

$$\mathbb{P}\left[|X_n - X| \xrightarrow[n \to \infty]{} 0\right] = 1, \quad \forall \varepsilon > 0.$$

a) We prove that almost sure convergence implies convergence in probability. To do so, we fix $\varepsilon > 0$ and consider the sequence of sets

$$\Omega_n := \bigcup_{m \geq n} \{\omega \,|\, |X_m(\omega) - X(\omega)| > \varepsilon\}.$$

i) Show that the sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ is decreasing towards the set

$$\Omega_\infty := \bigcap_{n \geq 1} \Omega_n.$$

ii) Show that

$$\lim_{n \to \infty} \mathbb{P}\left[|X_n - X| > \varepsilon\right] \leq \mathbb{P}\left[\Omega_\infty\right].$$

iii) Using that $X_n \overset{a.s.}{\to} X$, show that

$$\mathbb{P}\left[\Omega_\infty\right] = 0,$$

which concludes the proof.

b) Consider the probability space $([0,1], \mathcal{B}([0,1]), \mathbb{P})$, where $\mathbb{P}$ is the uniform distribution over $[0,1]$, and the sequence of random variables

$$X_n := \mathbb{1}_{\left[(n-2^{\lfloor \log_2(n) \rfloor})/2^{\lfloor \log_2(n) \rfloor},\ (n-2^{\lfloor \log_2(n) \rfloor}+1)/2^{\lfloor \log_2(n) \rfloor}\right]}, \quad n \geq 1.$$

Show that convergence in probability *does not* imply almost sure convergence.