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Mathematics of Information

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Solutions to Problem Set 1

Problem 1 Unconditional convergence.

The set $\mathcal{S} = \{x_k \mid k \in \mathbb{N}\}$ is orthogonal. Therefore, by Lemma 3 in the *Hilbert spaces* handout, the sum

$$\sum_{k=1}^{\infty} x_k$$

converges unconditionally if and only if

$$\sum_{k=1}^{\infty} \|x_k\|^2 < \infty.$$

Since

$$\sum_{k=1}^{\infty} \|x_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

it converges unconditionally.

Problem 2 Cauchy-Schwarz inequality.

a) We perform the expansion suggested by the hint to obtain

$$\begin{aligned} \Big\| \|y\|^2 x - \langle x, y\rangle y \Big\|^2 &= \Big\langle \|y\|^2 x - \langle x, y\rangle y, \|y\|^2 x - \langle x, y\rangle y \Big\rangle \\ &= \|y\|^4 \|x\|^2 + |\langle x, y\rangle|^2 \|y\|^2 - \langle \|y\|^2 x, \langle x, y\rangle y\rangle - \langle \langle x, y\rangle y, \|y\|^2 x\rangle \\ &= \|y\|^2 \Big\{ \|x\|^2 \|y\|^2 - |\langle x, y\rangle| \Big\}. \end{aligned} \tag{1}$$

The left-hand-side is always nonnegative and so is the right-hand-side. This implies the desired result $|\langle x, y\rangle| \leq \|x\| \|y\|$.

The equality case can be easily treated by observing that it corresponds to the case where (1) is vanishing. The left-hand-side must therefore vanish and $x = cy$ with $c := \langle x, y\rangle / \|y\|^2$. The next question provides an alternative proof.

b) Let us assume that $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = 1$. Then from

$$\langle x - y, y\rangle = \langle x, y\rangle - \|y\| = 0,$$

we conclude that $x-y$ is orthogonal to y. Applying Pythagoras theorem to $x=(x-y)+y$, we can write

$$\|x\|^2 = \|x-y\|^2 + \|y\|^2,$$

which implies that $\|x-y\|^2 = 0$ and hence $x = y$. Now, if $\|x\| \neq 1$ and $\|y\| \neq 1$, we can re-write $|\langle x, y\rangle| = \|x\|\|y\|$ as

$$\left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle\right| = 1$$

Let us define

$$\phi = \arg \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle.$$

We then have

$$\left\langle \frac{e^{-i\phi}x}{\|x\|}, \frac{y}{\|y\|} \right\rangle = 1,$$

and by defining $\tilde{x} = e^{-i\phi}x/\|x\|$ and $\tilde{y} = y/\|y\|$. We recover the first case, where $\|\tilde{x}\| = \|\tilde{y}\| = 1$ and $\langle \tilde{x}, \tilde{y}\rangle = 1$. We then have $\tilde{x} = \tilde{y}$, which gives

$$x = e^{i\phi}\frac{\|x\|}{\|y\|}y = cy,$$

where $c = e^{i\phi}\|x\|/\|y\|$.

Problem 3 A norm inequality.

The first inequality is obtained by taking the square root in the following inequality

$$\|x\|_2^2 = \sum_{i=1}^{n} \left|x_{(i)}\right|^2 \leq \sum_{i=1}^{n} \left|x_{(i)}\right|^2 + \sum_{i,j,i\neq j} \left|x_{(i)}\right| \left|x_{(j)}\right| = \left(\sum_{i=1}^{n} \left|x_{(i)}\right|\right)^2 = \|x\|_1^2,$$

and the second one follows by application of the Cauchy-Schwarz inequality according to

$$\|x\|_1 = \langle x, \mathrm{sgn}(x)\rangle \overset{\text{C.S.}}{\leq} \|x\|_2\|\mathrm{sgn}(x)\|_2 \leq \sqrt{n}\|x\|_2,$$

with $\mathrm{sgn}(x) \in \mathbb{R}^n$ defined as

$$\mathrm{sgn}(x)_{(i)} \coloneqq \begin{cases} -1, & \text{if } x_{(i)} < 0, \\ +1, & \text{if } x_{(i)} > 0, \\ 0, & \text{if } x_{(i)} = 0. \end{cases}$$

Problem 4 The spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

a) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \mathbb{1}_{(0,1]}(x)\frac{1}{\sqrt{x}}$. We have

$$\int_{\mathbb{R}} |f(x)|\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x = \left[2\sqrt{x}\,\right]_0^1 = 2 < \infty,$$

so $f \in L^1(\mathbb{R})$. Similarly,

$$\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x = \int_0^1 \frac{1}{x} \mathrm{d}x = [\log x\,]_0^1 = \infty,$$

so $f \notin L^2(\mathbb{R})$.

b) Let $f \in L^1(\mathbb{R})$ be an arbitrary function uniformly bounded by $A < \infty$. Then

$$\|f\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |f(x)| \cdot \underbrace{|f(x)|}_{\leq A} \mathrm{d}x \leq A \int_{\mathbb{R}} |f(x)| \mathrm{d}x = A\,\|f\|_{L^1(\mathbb{R})} < \infty,$$

Taking the square root of both sides yields the desired result.

c) For f vanishing outside of $[a, b]$, we have

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \mathrm{d}x = \int_a^b |f(x)| \cdot 1\, \mathrm{d}x.$$

Applying Cauchy-Schwartz inequality $\langle x, y\rangle_{L^2([a,b])} \leq \|x\|_{L^2([a,b])} \|y\|_{L^2([a,b])}$ with $x = |f|$ and $y = 1$, one gets

$$\|f\|_{L^1(\mathbb{R})} \leq \left(\int_a^b |f(x)|^2 \mathrm{d}x \right)^{1/2} \left(\int_a^b 1^2 \mathrm{d}x \right)^{1/2} = \|f\|_{L^2(\mathbb{R})} \sqrt{b-a} < \infty.$$

Problem 5 Projection on closed subspaces.

a) Let $y_1, y_2 \in \mathcal{S}$ such that

$$\|x - y_1\| = \|x - y_2\| = d \triangleq \min_{z \in \mathcal{S}} \|x - z\|.$$

The parallelogram law applied to $(x - y_1)$ and $(x - y_1)$ implies that

$$\begin{aligned} 4d^2 &= 2\|x - y_1\|^2 + 2\|x - y_2\|^2 \\ &= \|(x - y_1) + (x - y_2)\|^2 + \|(x - y_1) - (x - y_2)\|^2 \\ &= \|2x - y_1 - y_2\|^2 + \|y_2 - y_1\|^2 \\ &= 4\|x - (y_1 + y_2)/2\|^2 + \|y_2 - y_1\|^2 \\ &\geq 4d^2 + \|y_2 - y_1\|^2, \end{aligned}$$

where we used the fact that $(y_1 + y_2)/2 \in \mathcal{S}$ in the last step. Therefore, $\|y_2 - y_1\|^2 \leq 0$, which is only possible if $y_1 = y_2$.

b) Let $y_1, y_2 \in \mathcal{S}$ such that $(x - y_1) \in \mathcal{S}^\perp$ and $(x - y_2) \in \mathcal{S}^\perp$. Then, $y_1 - y_2 \in \mathcal{S}$, and, for $z \in \mathcal{S}$,

$$\begin{aligned} \langle y_1 - y_2, z\rangle &= \langle y_1, z\rangle - \langle y_2, z\rangle \\ &= \langle y_1, z\rangle - \langle y_2, z\rangle + \langle x, z\rangle - \langle x, z\rangle \\ &= \langle x - y_2, z\rangle - \langle x - y_1, z\rangle \\ &= 0. \end{aligned}$$

In particular, we obtain, for $z = y_1 - y_2$, that $\|y_1 - y_2\|^2 = \langle y_1 - y_2, y_1 - y_2 \rangle = 0$, which is only possible if $y_1 = y_2$.

Problem 6 A surjective linear isometry is unitary.

Let $\mathcal{H}$ be a real Hilbert space and $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ a surjective linear isometry. For $x, y \in \mathcal{H}$, we use the linearity of $\mathbb{T}$ and the polarization identity to obtain

$$\begin{aligned}
\langle \mathbb{T}x, \mathbb{T}y \rangle &= \frac{1}{4}\left(\|\mathbb{T}x + \mathbb{T}y\|^2 - \|\mathbb{T}x - \mathbb{T}y\|^2\right) \\
&= \frac{1}{4}\left(\|\mathbb{T}(x + y)\|^2 - \|\mathbb{T}(x - y)\|^2\right) \\
&= \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right) \\
&= \langle x, y \rangle.
\end{aligned}$$

Let $\mathbb{I}$ denote the identity operator on $\mathcal{H}$. Then, given any $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
0 &= \langle \mathbb{T}x, \mathbb{T}y \rangle - \langle x, y \rangle \\
&= \langle \mathbb{T}^*\mathbb{T}x, y \rangle - \langle \mathbb{I}x, y \rangle \\
&= \langle (\mathbb{T}^*\mathbb{T} - \mathbb{I})x, y \rangle.
\end{aligned}$$

By setting $y = (\mathbb{T}^*\mathbb{T} - \mathbb{I})x$, we obtain $\|(\mathbb{T}^*\mathbb{T} - \mathbb{I})x\|^2 = 0$, and thus $\mathbb{T}^*\mathbb{T}x = \mathbb{I}x$. Since x was arbitrary, this implies $\mathbb{T}^*\mathbb{T} = \mathbb{I}$. Now, if we can show that $\mathbb{T}$ is invertible, then we would have $\mathbb{T}^{-1} = \mathbb{I}\mathbb{T}^{-1} = \mathbb{T}^*\mathbb{T}\mathbb{T}^{-1} = \mathbb{T}^*$, and thus $\mathbb{T}$ would be unitary, as desired. Note that $\mathbb{T}$ is invertible if and only if $\mathbb{T}$ is both injective and surjective. We know that it is surjective, by assumption, so it remains to show that it is injective. Indeed, let $x \in \mathcal{H}$ be such that $\mathbb{T}x = 0$. Then $x = \mathbb{I}x = \mathbb{T}^*\mathbb{T}x = \mathbb{T}^*0 = 0$. Therefore $\ker(\mathbb{T}) = \{0\}$, so $\mathbb{T}$ is injective, as desired.

Problem 7 Continuty and Fourier Transform.

a) We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_1(x)|^2 \,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{(1 + |x|)^2} \,\mathrm{d}x \\
&= 2\int_0^{\infty} \frac{1}{(1 + x)^2} \mathrm{d}x \\
&= 2 \cdot \frac{-1}{1 + x}\bigg|_0^{\infty} = 2 < \infty,
\end{aligned}$$

so $f_1 \in L^2(\mathbb{R})$ with $\|f_1\|_{L^2(\mathbb{R})} = \sqrt{2}$. On the other hand,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_1(x)| \,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{1 + |x|} \,\mathrm{d}x \\
&\geq \int_1^{\infty} \frac{1}{1 + x} \,\mathrm{d}x \\
&= \log(1 + x)\Big|_1^{\infty} = \infty,
\end{aligned}$$

so $f_1 \notin L^1(\mathbb{R})$.

b) Let f_2 be the function given by $f_2(x) = \mathbb{1}_{(0,1]}(x)\frac{1}{\sqrt{x}}$, $x \in \mathbb{R}$. We then have

$$\int_{-\infty}^{\infty} |f_2(x)| \mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x = \frac{1}{2}\sqrt{x}\,\Big|_0^1 = \frac{1}{2} < \infty,$$

so $f_2 \in L^1(\mathbb{R})$. Moreover,

$$\int_{-\infty}^{\infty} |f_2(x)|^2 \,\mathrm{d}x = \int_0^1 \frac{1}{x} \,\mathrm{d}x = \log x \,|_0^1 = \infty,$$

so $f_2 \notin L^2(\mathbb{R})$.

c) i) As f is an element of $L^1(\mathbb{R})$, its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ is, indeed, defined. Furthermore, as f is also an element of $L^2(\mathbb{R})$, so is $\hat{f}$, by Plancherel's theorem. Therefore, applying the triangle inequality in the space $L^2(\mathbb{R})$, we have

$$\|G_f\|_{L^2(\mathbb{R})} = \left\| |\hat{f}| + |H_f| \right\|_{L^2(\mathbb{R})} \leq \|\hat{f}\|_{L^2(\mathbb{R})} + \|H_f\|_{L^2(\mathbb{R})} < \infty,$$

as $\hat{f} \in L^2(\mathbb{R})$ by the above and $H_f \in L^2(\mathbb{R})$ by assumption.

ii) We estimate

$$\begin{aligned}
\|\hat{f}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\hat{f}(\omega)| \,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \frac{1}{1+|\omega|} \cdot G_f(\omega) \,\mathrm{d}\omega \\
&\overset{CS}{\leq} \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\omega|)^2} \,\mathrm{d}\omega \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |G_f(\omega)|^2 \,\mathrm{d}\omega \right)^{\frac{1}{2}} \\
&= \sqrt{2}\,\|G_f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

As $G_f \in L^2(\mathbb{R})$ by (i) above, we have $\|\hat{f}\|_{L^1(\mathbb{R})} \leq \sqrt{2}\,\|G_f\|_{L^2(\mathbb{R})} < \infty$, and thus $\hat{f} \in L^1(\mathbb{R})$.

iii) Denote the Fourier transform of f by g, i.e., $g = \hat{f}$, and let f^-be the time-reversal of f, i.e., $f^-(x) = f(-x)$, $x \in \mathbb{R}$. We have shown in (ii) that $g \in L^1(\mathbb{R})$, so its Fourier transform $\hat{g}$ is defined, and additionally we know that $\hat{g}$ is continuous. Hence $\hat{g} = \hat{\hat{f}} = f^-$ is continuous. Thus, as the time-reversal of f is continuous, so is f itself.

Problem 8 Parallelogram law.

a) i) For $x, y \in H$ it holds that

$$\begin{aligned}
\|x+y\|_H^2 + \|x-y\|_H^2 &= \langle x+y, x+y\rangle_H + \langle x-y, x-y\rangle_H \\
&= \langle x,x\rangle_H + 2\langle x,y\rangle_H + \langle y,y\rangle_H \\
&\quad + \langle x,x\rangle_H - 2\langle x,y\rangle_H + \langle y,y\rangle_H \\
&= 2\langle x,x\rangle_H + 2\langle y,y\rangle_H \\
&= 2(\|x\|_H^2 + \|y\|_H^2).
\end{aligned}$$

ii) For $x_1, \dots, x_n \in H$ it holds that

$$
\begin{aligned}
&\sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|_H^2 \\
&= \sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \Big\langle \sum_{i=1}^n \varepsilon_i x_i, \sum_{j=1}^n \varepsilon_j x_j \Big\rangle_H \\
&= \sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \sum_{i=1}^n \sum_{j=1}^n \varepsilon_i \varepsilon_j \langle x_i, x_j \rangle_H \\
&= \sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \Big(\sum_{i=1}^n \varepsilon_i^2 \langle x_i, x_i \rangle_H + \sum_{\substack{1 \le i,j \le n \\ i \neq j}} \varepsilon_i \varepsilon_j \langle x_i, x_j \rangle_H \Big) \\
&= \sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \sum_{i=1}^n \|x_i\|_H^2 + \sum_{\substack{1 \le i,j \le n \\ i \neq j}} \Big(\sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \varepsilon_i \varepsilon_j \Big) \langle x_i, x_j \rangle_H \\
&= 2^n \sum_{i=1}^n \|x_i\|_H^2 .
\end{aligned}
$$

Here we used that, for any pair i, j with $i \neq j$, the four possibilities $\{(\varepsilon_i = +1, \varepsilon_j = +1), (\varepsilon_i = -1, \varepsilon_j = +1), (\varepsilon_i = +1, \varepsilon_j = -1), (\varepsilon_i = -1, \varepsilon_j = -1)\}$ occur exactly 2^{n-2} times. Therefore, the summand takes equaly likely the value $+1$ and -1, which implies

$$\sum_{(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,+1\}^n} \varepsilon_i \varepsilon_j = 0$$

b) i)
- $\langle x, x \rangle = \frac{1}{4}(\|x + x\|_E^2 - \|x - x\|_E^2) = \frac{1}{4}\|2x\|_E^2 = \|x\|_E^2$.
  The last equality follows from the absolute homogeneity of the norm $\|.\|_E$.
- $\langle x, x \rangle = 0 \implies \|x\|_E^2 = 0 \implies x = 0$. The last implication comes from the definiteness of the norm $\|.\|_E$.
- $\langle x, y \rangle = \frac{1}{4}(\|x + y\|_E^2 - \|x - y\|_E^2) = \frac{1}{4}(\|y + x\|_E^2 - \|y - x\|_E^2) = \langle y, x \rangle$. Here the absolute homogeneity of the norm $\|.\|_E$ has again been used.

ii) We first prove the identities in the hint:

$$
\begin{aligned}
&\|x_1 + x_2 + 2y\|_E^2 + \|x_1 - x_2\|_E^2 \\
&= \|(x_1 + y) + (x_2 + y)\|_E^2 + \|(x_1 + y) - (x_2 + y)\|_E^2 \\
&\overset{Par.}{=} 2\|x_1 + y\|_E^2 + 2\|x_2 + y\|_E^2,
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
&\|x_1 + x_2 + 2y\|_E^2 + \|x_1 + x_2\|_E^2 \\
&= \|(x_1 + x_2 + y) + y\|_E^2 + \|(x_1 + x_2 + y) - y\|_E^2 \\
&\overset{Par.Gl.}{=} 2\|x_1 + x_2 + y\|_E^2 + 2\|y\|_E^2.
\end{aligned}
\tag{3}
$$

The second group of identities follows from the same techniques used above. The proof of additivity now follows as

$$\begin{aligned}
\langle x_1 + x_2, y\rangle &= \frac{1}{4}(\|x_1 + x_2 + y\|_E^2 - \|x_1 + x_2 - y\|_E^2) \\
&= \frac{1}{8}\Big((2\|x_1 + x_2 + y\|_E^2 + 2\|y\|_E^2) - (2\|x_1 + x_2 - y\|_E^2 + 2\|y\|_E^2)\Big) \\
&\overset{(3)}{=} \frac{1}{8}\Big((\|x_1 + x_2 + 2y\|_E^2 + \|x_1 + x_2\|_E^2) \\
&\quad - (\|x_1 + x_2 - 2y\|_E^2 + \|x_1 + x_2\|_E^2)\Big) \\
&= \frac{1}{8}(\|x_1 + x_2 + 2y\|_E^2 - \|x_1 + x_2 - 2y\|_E^2) \\
&= \frac{1}{8}\Big((\|x_1 + x_2 + 2y\|_E^2 + \|x_1 - x_2\|_E^2) \\
&\quad - (\|x_1 + x_2 - 2y\|_E^2 + \|x_1 - x_2\|_E^2)\Big) \\
&\overset{(2)}{=} \frac{1}{8}(2\|x_1 + y\|_E^2 + 2\|x_2 + y\|_E^2 - 2\|x_1 - y\|_E^2 - 2\|x_2 - y\|_E^2) \\
&= \frac{1}{4}(\|x_1 + y\|_E^2 - \|x_1 - y\|_E^2) + \frac{1}{4}(\|x_2 + y\|_E^2 - \|x_2 - y\|_E^2) \\
&= \langle x_1, y\rangle + \langle x_2, y\rangle.
\end{aligned}$$

iii) Let $x, y \in E$. We can show by induction that $\langle nx, y\rangle = n\langle x, y\rangle$ holds for all $n \in \mathbb{N}$. Indeed, both hand sides equal 0 for $n = 0$, and if $\langle nx, y\rangle = n\langle x, y\rangle$ holds for $n \in \mathbb{N}$, we can use the additivity of $\langle\cdot,\cdot\rangle$ previously shown to write that

$$\langle (n+1)x, y\rangle = \langle nx, y\rangle + \langle x, y\rangle = (n+1)\langle x, y\rangle.$$

In addition, we have that

$$\begin{aligned}
\langle -x, y\rangle &= \frac{\|-x+y\|^2 - \|-x\|^2 - \|y\|^2}{2} = \frac{\|-x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \\
&= \frac{2\|x\|^2 - 2\|y\|^2 - \|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} = -\frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \\
&= -\langle x, y\rangle,
\end{aligned}$$

which implies that $\langle nx, y\rangle = n\langle x, y\rangle$ holds for all $n \in \mathbb{Z}$. For $r = p/q \in \mathbb{Q}$, we then have

$$q\langle rx, y\rangle = \langle px, y\rangle = p\langle x, y\rangle,$$

which amounts to $\langle rx, y\rangle = r\langle x, y\rangle$. The norm $\|\cdot\|$ is continuous on E (because it is Lipschitz continuous with Lipschitz constant 1). Hence, by density of $\mathbb{Q}$ in $\mathbb{R}$, we can conclude that

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$$

for all $\lambda \in \mathbb{R}$.

c) We can verify that any two elements of the Banach space $\ell^2(\mathbb{Z})$ satisfy the parallelogram law: for all sequences $u = \{u_k\}_{k\in\mathbb{Z}}$ and $v = \{v_k\}_{k\in\mathbb{Z}}$, we have

$$\begin{aligned}\|u+v\|_2^2 + \|u-v\|_2^2 &= \sum_{k=-\infty}^{+\infty} |u_k+v_k|^2 + \sum_{k=-\infty}^{+\infty} |u_k-v_k|^2 \\ &= \sum_{k=-\infty}^{+\infty} |u_k|^2 + |v_k^2| + 2u_kv_k + |u_k|^2 + |v_k^2| - 2u_kv_k \\ &= 2\|u\|_2^2 + 2\|v\|_2^2.\end{aligned}$$

From b), we can conclude that the norm defined on $\ell^2(\mathbb{Z})$ is induced by an inner product and that $\ell^2(\mathbb{Z})$ is hence a Hilbert space[1]. To show that $\ell^1(\mathbb{Z})$ is not an inner product space, we define the two sequences $u = \{u_k\}_{k\in\mathbb{Z}}$ and $v = \{v_k\}_{k\in\mathbb{Z}}$ such that

$$u_k = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad v_k = \begin{cases} 1, & k=1 \\ 0, & \text{otherwise.} \end{cases}$$

We have that $\|u+v\|_1 = \|u-v\|_1 = 2$ and $\|u\|_1 = \|v\|_1 = 1$. Consequently, the parallelogram law does not hold, implying that $\ell^1(\mathbb{Z})$ is not an inner product space (using a)), and thus, it is not a Hilbert space.

Problem 9 Short-time Fourier Transform.

a) Let F be a function in the space $L^2(\mathbb{R}^2)$. Then we have

$$\begin{aligned}\|\mathcal{T}_aF\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} |F(y,y-x)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y,y-x)|^2 \,\mathrm{d}x \right) \mathrm{d}y \\ &\overset{x\mapsto y-v}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y,y-(y-v))|^2 \,\mathrm{d}v \right) \mathrm{d}y \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y,v)|^2 \,\mathrm{d}y \right) \mathrm{d}v = \|F\|_{L^2(\mathbb{R}^2)}^2.\end{aligned}$$

Thus $\mathcal{T}_a$ is a well-defined map on $L^2(\mathbb{R}^2)$. Now let F and G be two $L^2(\mathbb{R}^2)$ functions. Then

$$\begin{aligned}\langle \mathcal{T}_aF, G\rangle &= \int_{\mathbb{R}^2} F(y,y-x)\overline{G(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\ &\overset{x\mapsto y-v}{=} \int_{\mathbb{R}^2} F(y,v)\overline{G(y-v,y)}\,\mathrm{d}v\,\mathrm{d}y \\ &= \langle F, \mathcal{T}_a^*G\rangle,\end{aligned}$$

where $\mathcal{T}_a^*G(u,v) = G(u-v,u)$. Now for any $F \in L^2(\mathbb{R}^2)$ and $(x,y)\in\mathbb{R}^2$ we have

$$\begin{cases} \mathcal{T}_a{}^*\mathcal{T}_aF(x,y) = (\mathcal{T}_aF)(x-y,x) = F(x,x-(x-y)) = F(x,y), \\ \mathcal{T}_a\mathcal{T}_a{}^*F(x,y) = (\mathcal{T}_a^*F)(y,y-x) = F(y-(y-x),y) = F(x,y). \end{cases}$$

Therefore $\mathcal{T}_a{}^*\mathcal{T}_a = \mathcal{T}_a\mathcal{T}_a{}^* = \mathrm{Id}$ and hence $\mathcal{T}_a$ is invertible with inverse $\mathcal{T}_a{}^*$.

[1] We showed the proposition for a real vector space, but it also holds for a complex vector space.

b) Take any $f, g \in L^2(\mathbb{R})$. First note that $f \otimes \overline{g} \in L^2(\mathbb{R}^2)$, and thus also $\mathcal{T}_a(f \otimes \overline{g}) \in L^2(\mathbb{R}^2)$. Therefore $\mathcal{T}_a(f \otimes \overline{g})$ lies in the domain of $\mathcal{F}_2$ and so $\mathcal{F}_2\mathcal{T}_a(f \otimes \overline{g})$ is well defined. Now note that $(\mathcal{T}_a(f \otimes \overline{g}))(x,t) = f(t)\overline{g(t-x)}$. Applying the Cauchy-Schwarz inequality for an arbitrary, but fixed $x \in \mathbb{R}$, yields

$$\begin{aligned}\int_{\mathbb{R}} |f(t)\overline{g(t-x)}|\mathrm{d}t &= \langle |f|, |g(\cdot - x)|\rangle \\ &\leq \|f\|_{L^2(\mathbb{R})}\|g(\cdot - x)\|_{L^2(\mathbb{R})} \\ &= \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} < \infty,\end{aligned}$$

where in the last step we used $f, g \in L^2(\mathbb{R})$. Therefore $(\mathcal{T}_a(f \otimes \overline{g}))(x, \cdot) \in L^1(\mathbb{R})$, for all $x \in \mathbb{R}$, and thus the partial Fourier transform formula applies to $\mathcal{T}_a(f \otimes \overline{g})$. Finally, for any $(x, \omega) \in \mathbb{R}^2$ we have

$$\begin{aligned}\mathcal{F}_2\mathcal{T}_a(f \otimes \overline{g})(x,\omega) &= \int_{\mathbb{R}} (\mathcal{T}_a(f \otimes \overline{g}))(x,t)e^{-2\pi i\omega t}\mathrm{d}t \\ &= \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}\mathrm{d}t = (V_g f)(x,\omega),\end{aligned}$$

as desired.

c) Now, $V_g f \in L^2(\mathbb{R}^2)$ for $f, g \in L^2(\mathbb{R})$ follows since $f \otimes \overline{g} \in L^2(\mathbb{R}^2)$ for such f and g, and $\mathcal{T}_a$ and $\mathcal{F}_2$ are well-defined operators mapping $L^2(\mathbb{R}^2)$ functions to $L^2(\mathbb{R}^2)$ functions. Since $\mathcal{T}_a$ and $\mathcal{F}_2$ are unitary, we have

$$\begin{aligned}\langle V_{g_1}f_1, V_{g_2}f_2\rangle &= \langle \mathcal{F}_2\mathcal{T}_a(f_1 \otimes \overline{g_1}), \mathcal{F}_2\mathcal{T}_a(f_2 \otimes \overline{g_2})\rangle \\ &= \langle \mathcal{T}_a(f_1 \otimes \overline{g_1}), \mathcal{T}_a(f_2 \otimes \overline{g_2})\rangle \\ &= \langle f_1 \otimes \overline{g_1}, f_2 \otimes \overline{g_2}\rangle \\ &= \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle},\end{aligned}$$

for all $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$.