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Mathematics of Information

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Problem Set 2

Problem 1 Overcomplete expansion in $\mathbb{R}^2$. ☕

Consider the following example discussed in class. For the vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

we found that any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented according to

$$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}_2 \rangle \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}_3 \rangle \mathbf{e}_3$$

where

$$\tilde{\mathbf{e}}_1 = 2\mathbf{e}_1, \quad \tilde{\mathbf{e}}_2 = -\mathbf{e}_3, \quad \tilde{\mathbf{e}}_3 = -\mathbf{e}_1.$$

a) Find another set of vectors $\tilde{\mathbf{e}}'_1, \tilde{\mathbf{e}}'_2, \tilde{\mathbf{e}}'_3$, neither of which is collinear to neither of the vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ and such that any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as

$$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}'_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}'_2 \rangle \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}'_3 \rangle \mathbf{e}_3.$$

<u>Hint</u>: *Look for another right-inverse of the matrix*

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix}.$$

b) Now consider the vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \quad \tilde{\mathbf{e}}_1 = \begin{bmatrix}1\\-1\end{bmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix},$$

and show that any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \tilde{\mathbf{e}}_2.$$

Is it possible to find two vectors $\mathbf{e}'_1, \mathbf{e}'_2$, neither of which is collinear to neither of the vectors $\mathbf{e}_1, \mathbf{e}_2$ such that

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}'_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}'_2 \rangle \tilde{\mathbf{e}}_2?$$

Problem 2 Change of basis matrix between ONBs is unitary.

Let $\mathcal{B}_1 = \{g_n\}_{n=1}^N$ and $\mathcal{B}_2 = \{h_n\}_{n=1}^N$ be orthonormal bases for the Hilbert space $\mathbb{C}^N$. Show that the matrix $U \in \mathbb{C}^{N\times N}$ whose entries are given by $U_{jk} = \langle h_k, g_j \rangle_{\mathbb{C}^N}$ is unitary.

Problem 3 Examples of frames.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Determine for each of the following sets whether it is a frame for $\mathcal{H}$ or not. For sets that are a frame, determine the tightest possible frame bounds A, B, else prove that the set is not a frame.

a)

$$\{h_k\}_{k\in\mathbb{N}} = \{(-1)^k e_k\}_{k\in\mathbb{N}} = \{-e_1, e_2, -e_3, e_4, \dots\}$$

b)

$$\{h_k\}_{k\in\mathbb{N}} = \left\{e_1, \frac{1}{2}\,e_2, \frac{1}{2}\,e_2, \frac{1}{3}\,e_3, \frac{1}{3}\,e_3, \frac{1}{3}\,e_3, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \frac{1}{4}\,e_4, \dots\right\}$$

Problem 4 Local averaging operator.

Define the following local-averaging operator

$$(\mathcal{A}x)_n = \int_{n-1/2}^{n+1/2} x(t)\mathrm{d}t, \quad n \in \mathbb{Z},$$

that takes in a function $x \in L^2(\mathbb{R})$ and yields a sequence $\{(\mathcal{A}x)_n\}_{n\in\mathbb{Z}}$ of local averages.

a) Verify that $\mathcal{A}$ is a bounded linear operator from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z})$ and compute the adjoint $\mathcal{A}^* : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$ of $\mathcal{A}$.

b) Show that $\|\mathcal{A}^* y\|_{L^2(\mathbb{R})} = \|y\|_{\ell^2(\mathbb{Z})}$ for all $y \in \ell^2(\mathbb{Z})$.

c) Define $\mathrm{Im}(\mathcal{A}^*) = \{\mathcal{A}^* y : y \in \ell^2(\mathbb{Z})\}$. You may use - without proof - that $\mathrm{Im}(\mathcal{A}^*)$ is a closed subspace of $L^2(\mathbb{R})$, and thus a Hilbert space in its own right. For each $n \in \mathbb{Z}$ let $e_n = \mathbb{1}_{[n-1/2,\, n+1/2]}$ be the indicator function of the interval $[n - 1/2, n + 1/2]$.

- Show that $\mathcal{G} := \{e_n : n \in \mathbb{Z}\}$ is a frame for $\mathrm{Im}(\mathcal{A}^*)$.
- Show that $\mathcal{A}$ can be interpreted as the analysis operator associated with the frame $\mathcal{G}$.

Problem 5 Tight frames.

Let $\{f_k\}_{k=0}^\infty$ be a frame for the Hilbert space $\mathcal{H}$. Show that the following statements are equivalent:

a) $\{f_k\}_{k=0}^\infty$ is tight,

b) $\{f_k\}_{k=0}^\infty$ has a dual of the form $g_k = Cf_k$ for some constant $C > 0$.

Problem 6 Unitary transformation of a frame.

Let $\{f_k\}_{k\in\mathcal{K}}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds A and B. Let $\mathbb{U} : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Show that the set $\{\mathbb{U}f_k\}_{k\in\mathcal{K}}$ is again a frame for $\mathcal{H}$ and compute the corresponding frame bounds.

Problem 7 Complete set, but not a Frame.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Define the set

$$\{g_k\}_{k\in\mathbb{N}} = \{e_k + e_{k+1}\}_{k\in\mathbb{N}} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, \dots\}.$$

a) Show that the set $\{g_k\}_{k\in\mathbb{N}}$ is complete for $\mathcal{H}$.
Hint: Recall that $\|x\| < \infty$*, for all* $x \in \mathcal{H}$.

b) Show that the set $\{g_k\}_{k\in\mathbb{N}}$ is not a frame for $\mathcal{H}$.
Hint: It may be helpful to consider signals of the form $x_q = \sum_{\ell=1}^{\infty}(-q)^{\ell-1}e_\ell$ *with* $q \in (0, 1)$.