ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Chair for Mathematical Information Science

Prof. Dr. H. Bölcskei
Sternwartstrasse 7
CH-8092 Zürich

Mathematics of Information

Spring semester 2022

Solutions to Problem Set 2

Problem 1 Overcomplete expansion in $\mathbb{R}^2$.

a) Consider the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Our goal is to find vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ such that

$$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}_1 \rangle \, \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}_2 \rangle \, \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}_3 \rangle \, \mathbf{e}_3 = \underbrace{\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \tilde{\mathbf{e}}_1^T \\ \tilde{\mathbf{e}}_2^T \\ \tilde{\mathbf{e}}_3^T \end{bmatrix} \mathbf{x}.$$

In order to find these vectors, we are looking for a right inverse of the matrix $\mathbf{A}$. One possible right inverse can be found by noting that

$$\mathbf{A} \underbrace{\mathbf{A}^T (\mathbf{A}\mathbf{A}^T)^{-1}}_{\text{right inverse}} = \mathbf{I}.$$

First we calculate

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and the inverse

$$(\mathbf{A}\mathbf{A}^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and finally

$$\begin{bmatrix} \tilde{\mathbf{e}}_1'^T \\ \tilde{\mathbf{e}}_2'^T \\ \tilde{\mathbf{e}}_3'^T \end{bmatrix} = \mathbf{A}^T (\mathbf{A}\mathbf{A}^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

The vectors $\tilde{\mathbf{e}}_1', \tilde{\mathbf{e}}_2', \tilde{\mathbf{e}}_3'$ are given by

$$\tilde{\mathbf{e}}_1' = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \, \tilde{\mathbf{e}}_2' = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \, \tilde{\mathbf{e}}_3' = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}.$$

Comparing to the given set of vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ we find

$$\tilde{\mathbf{e}}_1 = 2\mathbf{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_2 = -\mathbf{e}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{e}}_3 = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

It should be emphasized that the right inverse is not unique: the system of equations

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}}_{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has infinitely many solutions of the form

$$\mathbf{B} = \begin{bmatrix} 1-\lambda & -\gamma \\ \lambda & 1+\gamma \\ \lambda & \gamma \end{bmatrix}$$

for any $\lambda, \gamma \in \mathbb{R}$. Any such matrix $\mathbf{B}$ is a valid right inverse of $\mathbf{A}$, which generates in general a different set of vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$.

b) We have

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \tilde{\mathbf{e}}_2 = \begin{bmatrix} \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{bmatrix} \mathbf{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\mathbf{B}} \mathbf{x},$$

and we easily verify

$$\begin{bmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \mathbf{I}.$$

Since $\mathbf{A}$ is a square matrix, the inverse is unique. Hence, there is no other matrix $\mathbf{B}$ satisfying the above equation. Thus, we can not find $\mathbf{e}_1', \mathbf{e}_2'$.

Problem 2 Change of basis matrix between ONBs is unitary.

We have to verify that $U^*U = UU^* = \mathrm{Id}$. First note that, for $j, k \in [N]$, we have

$$\begin{aligned} \langle h_k, h_j \rangle &= \left\langle \sum_{\ell=1}^{N} \langle h_k, g_\ell \rangle g_\ell, \sum_{\ell'=1}^{N} \langle h_j, g_{\ell'} \rangle g_{\ell'} \right\rangle = \sum_{\ell=1}^{N} \sum_{\ell'=1}^{N} \langle h_k, g_\ell \rangle \overline{\langle h_j, g_{\ell'} \rangle} \underbrace{\langle g_\ell, g_{\ell'} \rangle}_{=\delta_{\ell\ell'}} \\ &= \sum_{\ell=1} \langle h_k, g_\ell \rangle \overline{\langle h_j, g_\ell \rangle}. \end{aligned}$$

Therefore, for $j, k \in [N]$, we have

$$[U^*U]_{jk} = \sum_{\ell=1}^{N} [U^*]_{j\ell} U_{\ell k} = \sum_{\ell=1}^{N} \overline{U_{\ell j}} U_{\ell k} = \sum_{\ell=1}^{N} \overline{\langle h_j, g_\ell \rangle} \langle h_k, g_\ell \rangle = \langle h_k, h_j \rangle = \delta_{jk},$$

and so $U^*U = \mathrm{Id}$. A completely analogous computation shows that $UU^* = \mathrm{Id}$.

Problem 3 Examples of frames.

a) For arbitrary $x \in \mathcal{H}$, we compute

$$\sum_{k=1}^{\infty} | \langle x, h_k \rangle |^2 = \sum_{k=1}^{\infty} | \left\langle x, (-1)^k e_k \right\rangle |^2 = \sum_{k=1}^{\infty} | \langle x, e_k \rangle |^2 = \|x\|^2,$$

where the last equality holds because $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis (ONB) for $\mathcal{H}$ by assumption. This establishes that the set $\{h_k\}_{k\in\mathbb{N}}$ is a tight frame with frame bounds $A = B = 1$.

b) For arbitrary $x \in \mathcal{H}$, we compute

$$\sum_{k=1}^{\infty} | \langle x, h_k \rangle |^2 = \sum_{k=1}^{\infty} k \left| \frac{1}{k} \langle x, e_k \rangle \right|^2 = \sum_{k=1}^{\infty} \frac{1}{k} | \langle x, e_k \rangle |^2. \tag{1}$$

Next, we assume towards a contradiction that there exists a lower frame bound $A > 0$, i.e., $A\|x\|^2 \leq \sum_{k=1}^{\infty} | \langle x, h_k \rangle |^2, \forall x \in \mathcal{H}$. Fix an integer N such that $\frac{1}{N} < A$ and evaluate (1) for $x = e_N \in \mathcal{H}$ to get

$$\sum_{k=1}^{\infty} | \langle e_N, h_k \rangle |^2 = \frac{1}{N} |\langle e_N, e_N \rangle|^2 = \frac{1}{N} \|e_N\|_2 < A\|e_N\|^2.$$

This stands in contradiction to the assumption that A is a lower frame bound. As A was arbitrary no lower frame bound can therefore exist and $\{h_k\}_{k\in\mathbb{N}}$ is thus not a frame.

Problem 4 Local averaging operator.

a) Let $e_n = \mathbb{1}_{[n-1/2,\, n+1/2]}$ be the indicator function of the interval $[n - 1/2, n + 1/2]$. Note that $\|e_n\|_{L^2(\mathbb{R})} = 1$, and so $e_n \in L^2(\mathbb{R})$. Then, by the Cauchy-Schwarz inequality we have

$$\left| \int_{n-1/2}^{n+1/2} x(t)\, \mathrm{d}t \right| = |\langle e_n, x \rangle| \leq \|e_n\|_{L^2(\mathbb{R})} \|x\|_{L^2(\mathbb{R})} = \|x\|_{L^2(\mathbb{R})},$$

for any $x \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$, and thus $(\mathcal{A}x)_n$ is well-defined for all $x \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$. To show that $\mathcal{A}x \in \ell^2(\mathbb{Z})$ whenever $x \in L^2(\mathbb{R})$, we again use the Cauchy-Schwarz inequality to get

$$\begin{aligned} \|\mathcal{A}x\|^2_{\ell^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} |(\mathcal{A}x)_n|^2 = \sum_{n\in\mathbb{Z}^2} \left| \int_{n-1/2}^{n+1/2} x(t)\, \mathrm{d}t \right|^2 \\ &\leq \sum_{n\in\mathbb{Z}^2} \int_{n-1/2}^{n+1/2} |x(t)|^2\, \mathrm{d}t = \int_{\mathbb{R}} |x(t)|^2\, \mathrm{d}t = \|x\|^2_{L^2(\mathbb{R})}, \end{aligned}$$

for all $x \in L^2(\mathbb{R})$. In particular, we have $\|\mathcal{A}x\|_{\ell^2(\mathbb{Z})} < \infty$, and so $\mathcal{A}$ is well-defined. Also, $\mathcal{A}$ is linear, because integration is a linear operation. Finally, since $\|\mathcal{A}x\|^2_{\ell^2(\mathbb{Z})} \leq \|x\|_{L^2(\mathbb{R})}$, for all $x \in L^2(\mathbb{R})$, we have that $\mathcal{A}$ is bounded.

By definition of adjoint operators, $\mathcal{A}^*$ is the unique operator such that

$$\langle \mathcal{A}x, y \rangle_{\ell^2(\mathbb{Z})} = \langle x, \mathcal{A}^* y \rangle_{L^2(\mathbb{R})},$$

for all $x \in L^2(\mathbb{R})$ and $y \in \ell^2(\mathbb{Z})$. For arbitrary, but fixed x and y, we calculate

$$\begin{aligned}
\langle \mathcal{A}x, y\rangle_{\ell^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} (\mathcal{A}x)_n \cdot \overline{y_n} = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\,\mathrm{d}t \cdot \overline{y_n} \\
&= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\,\overline{y_n}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} x(t)(\mathcal{A}^*y)(t)\,\mathrm{d}y, \qquad (2)
\end{aligned}$$

where $\mathcal{A}^*y$ is the piecewise-constant function given by $(\mathcal{A}^*y)(t) = y_n$ for $t \in [n-\frac{1}{2}, n+\frac{1}{2})$.

b) It follows from the (a) that

$$\|\mathcal{A}^*y\|^2_{L^2(\mathbb{R})} = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} |y_n|^2\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} |y_n|^2 = \|y\|^2_{\ell^2(\mathbb{Z})} \quad \text{for all } y \in \ell^2(\mathbb{Z}),$$

as desired.

c) Let $\{e_n\}_{n\in\mathbb{Z}}$ be the indicator functions defined at the beginning of (a). To show that $\mathcal{G} = \{e_n : n \in \mathbb{Z}\}$ is a frame for $\operatorname{Im}(\mathcal{A}^*)$, take an arbitrary $x = \mathcal{A}^*y \in \operatorname{Im}(\mathcal{A}^*)$. Then, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\langle x, e_n\rangle|^2 &= \sum_{n\in\mathbb{Z}} |\langle \mathcal{A}^*y, e_n\rangle|^2 = \sum_{n\in\mathbb{Z}} \left| \int_{n-1/2}^{n+1/2} y_n\,\mathrm{d}t \right|^2 \\
&= \|y\|^2_{\ell^2(\mathbb{Z})} = \|\mathcal{A}^*y\|^2_{L^2(\mathbb{R})} = \|x\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

and so $\mathcal{G}$ is a (tight) frame for $\operatorname{Im}(\mathcal{A}^*)$. Thus we have

$$\mathcal{A}x = \{\langle x, e_n\rangle : n \in \mathbb{Z}\},$$

for all $x \in \operatorname{Im}(\mathcal{A}^*) \subset L^2(\mathbb{R})$, and so $\mathcal{A}$ is the analysis operator associated with the frame $\mathcal{G}$ for the Hilbert space $\operatorname{Im}(\mathcal{A}^*)$.

Problem 5 Tight frames.

a) Assume that $\{f_k\}_{k=0}^{\infty}$ is tight. Then there exists a constant $A > 0$ such that

$$\sum_{k=0}^{\infty} |\langle f, f_k\rangle|^2 = A\|f\|^2$$

for all $f \in \mathcal{H}$. We can define $g_k = A^{-1}f_k$ for all $k \in \mathbb{N}$. We have then

$$\sum_{k=0}^{\infty} \langle f, g_k\rangle f_k = \sum_{k=0}^{\infty} \langle f, f_k\rangle g_k = A^{-1} \underbrace{\sum_{k=0}^{\infty} \langle f, f_k\rangle f_k}_{=\mathbb{S}f = Af} = f,$$

where we used the fact that the frame operator $\mathbb{S}$ satisfies $\mathbb{S} = A\mathbb{I}$ since $\{f_k\}_{k=0}^{\infty}$ is a tight frame with frame bound A. Therefore, $\{g_k\}_{k=0}^{\infty}$ forms a dual frame[1] of $\{f_k\}_{k=0}^{\infty}$.

b) Conversely, assume that $\{f_k\}_{k=0}^{\infty}$ has a dual of the form $g_k = Cf_k$ with $C > 0$. Then for all $f \in \mathcal{H}$, we have

$$f = \sum_{k=0}^{\infty}\langle f, g_k\rangle f_k = \sum_{k=0}^{\infty}\langle f, f_k\rangle g_k = C\sum_{k=0}^{\infty}\langle f, f_k\rangle f_k = C\mathbb{S}f,$$

which shows that the frame operator is $\mathbb{S} = C^{-1}\mathbb{I}$, and that $\{f_k\}_{k=0}^{\infty}$ is hence a tight frame.

Problem 6 Unitary transformation of a frame.

Since $\{f_k\}_{k\in\mathcal{K}}$ is a frame with frame bounds A and B, we have

$$A\|f\|^2 \leq \sum_{k\in\mathcal{K}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2.$$

Moreover, since $\mathbb{U}$ is a unitary operator, one has $\mathbb{U}^*\mathbb{U} = \mathbb{U}\mathbb{U}^* = \mathbb{I}$. Hence, we have

$$\|\mathbb{U}^*f\|^2 = \langle \mathbb{U}^*f, \mathbb{U}^*f\rangle = \langle \mathbb{U}\mathbb{U}^*f, f\rangle = \langle f, f\rangle = \|f\|^2.$$

We have then

$$\sum_{k\in\mathcal{K}} |\langle f, \mathbb{U}f_k\rangle|^2 = \sum_{k\in\mathcal{K}} |\langle \mathbb{U}^*f, f_k\rangle|^2 \leq B\|\mathbb{U}^*f\|^2 = B\|f\|^2,$$

which establishes the upper frame bound. Next,

$$\sum_{k\in\mathcal{K}} |\langle f, \mathbb{U}f_k\rangle|^2 = \sum_{j\in\mathcal{K}} |\langle \mathbb{U}^*f, f_k\rangle|^2 \geq A\|\mathbb{U}^*f\|^2 = A\|f\|^2,$$

which establishes the lower frame bound. Therefore, $\{\mathbb{U}f_k\}_{k\in\mathcal{K}}$ is a frame for $\mathcal{H}$ with the same frame bounds as $\{f_k\}_{k\in\mathcal{K}}$.

Problem 7 Complete but not a Frame.

a) We prove that $\{g_k\}_{k\in\mathbb{N}}$ is complete by showing that the only signal $x \in \mathcal{H}$ that satisfies $\langle x, g_k\rangle = 0$, $\forall k \in \mathbb{N}$, is $x = 0$. Take $x \in \mathcal{H}$ with $0 = \langle x, g_k\rangle = \langle x, e_k + e_{k+1}\rangle$, $\forall k \in \mathbb{N}$. Hence,

$$\langle x, e_k\rangle = -\langle x, e_{k+1}\rangle\,,\ \forall k \in \mathbb{N},$$

which implies $|\langle x, e_k\rangle| = C$, $\forall k \in \mathbb{N}$, for some $C \geq 0$. Further, owing to $x \in \mathcal{H}$, we have $\|x\| < \infty$ and thus

$$\infty > \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 = \sum_{k=1}^{\infty} C^2, \tag{3}$$

where we used that $\{e_k\}_{k\in\mathbb{N}}$ is an ONB. The proof is concluded by noting that (3) can hold only if $C = 0$ and thus $x = 0$.

b) We prove that $\{g_k\}_{k\in\mathbb{N}}$ is not a frame by showing that no lower frame bound $A > 0$ exists. To this end, we fix $q \in (0, 1)$, consider the signal $x_q = \sum_{\ell=1}^{\infty}(-q)^{\ell-1}e_\ell$ and start by showing

[1] Note that $\{g_k\}_{k=0}^{\infty}$ is in fact the canonical dual frame of $\{f_k\}_{k=0}^{\infty}$, since $g_k = \mathbb{S}^{-1}f_k$ for all $k \in \mathbb{N}$.

that $x_q \in \mathcal{H}$. Since $\{e_k\}_{k\in\mathbb{N}}$ is an ONB by assumption, we can write

$$\|x_q\|^2 = \sum_{k=1}^{\infty} |\langle x_q, e_k\rangle|^2 = \sum_{k=1}^{\infty} (q^2)^{k-1} = \frac{1}{1-q^2} < \infty, \quad \forall q \in (0,1),$$

which establishes that $x_q \in \mathcal{H}$. Next, we compute

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle x_q, g_k\rangle|^2 &= \sum_{k=1}^{\infty} \left|\left\langle \sum_{\ell=1}^{\infty} (-q)^{\ell-1} e_\ell, e_k + e_{k+1} \right\rangle\right|^2 \\
&= \sum_{k=1}^{\infty} |(-q)^{k-1} + (-q)^k|^2 \\
&= \sum_{k=1}^{\infty} |(1-q)(-q)^{k-1}|^2 \\
&= (1-q)^2 \sum_{k=1}^{\infty} (q^2)^{k-1} \\
&= (1-q)^2 \frac{1}{1-q^2} = (1-q)^2 \|x_q\|^2.
\end{aligned}$$

The equality $\sum_{k=1}^{\infty} |\langle x_q, g_k\rangle|^2 = (1-q)^2\|x_q\|^2$ then establishes that there can be no $A > 0$ such that $A\|x\|^2 \leq \sum_{k=1}^{\infty} |\langle x, g_k\rangle|^2$, $\forall x \in \mathcal{H}$, as we can always find a $q \in (0,1)$ so that $(1-q)^2 < A$.