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Mathematics of Information

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Problem Set 3

Problem 1 Unitary operators. ☕

Let $U \in \mathbb{C}^{N\times N}$ for some $N \in \mathbb{N}$.

a) Show that $U^H U = \mathrm{Id}$ implies $UU^H = \mathrm{Id}$.

Now consider the operator $\mathcal{A} : \ell_2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as

$$(\mathcal{A}x_n)(t) = \sum_{n\in\mathbb{Z}} x_n e_n(t),$$

and it's adjoint $\mathcal{A}^* : L^2(\mathbb{R}) \to \ell_2(\mathbb{R})$ given by

$$(\mathcal{A}^* y)_n = \left(\int_{n-1/2}^{n+1/2} y(t)\mathrm{d}t\right)_n = \left(\langle y, e_n\rangle\right)_n, \quad n \in \mathbb{Z},$$

where

$$e_n(t) = \begin{cases} 1 & \text{if } n - \frac{1}{2} < t < n + \frac{1}{2} \\ 0 & \text{else} \end{cases}.$$

b) Show that $\mathcal{A}^*\mathcal{A} = \mathrm{Id}$.

c) Show that $\mathcal{A}\mathcal{A}^* \neq \mathrm{Id}$.

d) Reconsider your proof from subtask (a). Why does the same argument not work to show that $\mathcal{A}^*\mathcal{A} = \mathrm{Id}$ implies $\mathcal{A}\mathcal{A}^* = \mathrm{Id}$?

Problem 2 Redundancy of a frame. ☕☕

Let $\{\mathbf{f}_k\}_{k=1}^N$ be a frame for $\mathbb{C}^M$ with $N > M$. Assume that the frame elements are normalized such that $\|\mathbf{f}_k\| = 1$ for all k. The ratio N/M is called redundancy of the frame.

a) Assume that $\{\mathbf{f}_k\}_{k=1}^N$ is a tight frame with frame bound A. Show that $A = N/M$.

b) Now assume that A and B are lower and upper frame bounds of $\{\mathbf{f}_k\}_{k=1}^N$, respectively. Show that $A \leq N/M \leq B$.

Problem 3 Frame expansion with noise. ☕

Let $\{\mathbf{g}_j\}_{j=1}^M$ be a tight frame for $\mathbb{C}^N$ $(N \leq M)$ with $\|\mathbf{g}_j\| = 1$ for all $1 \leq j \leq M$. We know that every $\mathbf{f} \in \mathbb{C}^N$ can be perfectly reconstructed from its frame expansion coefficients according to

$$\mathbf{f} = \frac{1}{A} \sum_{j=1}^{M} \langle \mathbf{f}, \mathbf{g}_j \rangle \, \mathbf{g}_j,$$

where $A = M/N$. Now, assume that the frame expansion coefficients are subject to noise:

$$\langle \mathbf{f}, \mathbf{g}_j \rangle \to \langle \mathbf{f}, \mathbf{g}_j \rangle + w_j$$

where $\{w_j\}_{j=1}^M$ are independent, zero-mean random variables with variance N_0 each. After reconstruction, we obtain in this case

$$\mathbf{f}_{\mathrm{w}} = \frac{1}{A} \sum_{j=1}^{M} (\langle \mathbf{f}, \mathbf{g}_j \rangle + w_j) \mathbf{g}_j.$$

Compute the mean squared error (MSE) of the noisy reconstruction, defined as $\mathbb{E}\{\|\mathbf{f} - \mathbf{f}_{\mathrm{w}}\|^2\}$. How does the MSE depend on the redundancy $r = M/N$? Give an interpretation of the result.

Problem 4 DFT as a signal expansion. ☕☕

The Discrete Fourier Transform (DFT) of an N-point signal $f(n)$, $n = 0, 1, \ldots, N-1$, is defined as

$$\widehat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{k}{N} n}$$

Find the corresponding inverse transform and show that the DFT can be interpreted as a signal expansion in $\mathbb{C}^N$.

Problem 5 Weyl-Heisenberg systems. ☕☕☕

For this problem we use the two-indices notation $f_{x,y}$ to denote time-frequency shifts, that is, if $f : \mathbb{R} \to \mathbb{C}$ is a function and $x, y \in \mathbb{R}$, then we shall write $f_{x,y}(t) = e^{2\pi i y t} f(t - x)$.

You may use — without proof — that for all $f, h \in L^2(\mathbb{R})$ the function $(x, y) \mapsto \langle f, h_{x,y} \rangle$ is an $L^2(\mathbb{R}^2)$ function, that is

$$\int_{\mathbb{R}^2} |\langle f, h_{x,y} \rangle|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

Furthermore, you may also use — without proof — the following identity:

$$\int_{\mathbb{R}^2} \langle f, g_{x,y} \rangle \overline{\langle u, v_{x,y} \rangle} \, \mathrm{d}x \, \mathrm{d}y = \langle f, u \rangle \overline{\langle g, v \rangle} \quad \text{for all } f, g, u, v \in L^2(\mathbb{R}). \tag{IR}$$

a) Consider a Weyl-Heisenberg system $\mathcal{G} = \{g_{mT,nF}\}_{m,n \in \mathbb{Z}}$ with time-frequency parameters $T > 0$ and $F > 0$. Assume that $\mathcal{G}$ is a frame for $L^2(\mathbb{R})$. Let $\mathbb{S}$ be the corresponding frame operator and $\tilde{g} = \mathbb{S}^{-1} g$ the canonical dual function. We know that $\tilde{\mathcal{G}} = \{\tilde{g}_{mT,nF}\}_{m,n \in \mathbb{Z}}$ is the canonical dual frame to $\mathcal{G}$, and that the following reconstruction formula holds:

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT,nF}\rangle g_{mT,nF} \quad \text{for all } f \in L^2(\mathbb{R}).$$

Using this reconstruction formula, prove that

$$\langle f, h\rangle = \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \tag{1}$$

for all $f, h \in L^2(\mathbb{R})$ and all $x, y \in \mathbb{R}$.
[Hint: Expand $f_{-x,-y}$ using the reconstruction formula, and then take the inner product of both sides with $h_{-x,-y}$.]

b) By integrating both sides of (1) over $(x, y) \in [0, T) \times [0, F)$ for a fixed pair of functions f and h, show that $\langle g, \tilde{g}\rangle = TF$. Justify the validity of any manipulations you do by verifying the following absolute convergence property:

$$\sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T |\langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle| \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

[Hint: For both calculations, rewrite the resulting expression as an integral over $\mathbb{R}^2$ which does not involve infinite summation.]

c) Assume the following result from the lectures without proof:

Lemma 1 *Denote by $\mathcal{K}$ a countable index set. Let $\{h_k\}_{k\in\mathcal{K}}$ be a frame for a Hilbert space $\mathcal{H}$ and $\{\tilde{h}_k\}_{k\in\mathcal{K}}$ its canonical dual frame. For a fixed $z \in \mathcal{H}$, let $c_k = \langle z, \tilde{h}_k\rangle$ so that $z = \sum_{k\in\mathcal{K}} c_k h_k$. If it is possible to find scalars $\{a_k\}_{k\in\mathcal{K}}$ such that $z = \sum_{k\in\mathcal{K}} a_k h_k$, then we must have*

$$\sum_{k\in\mathcal{K}} |a_k|^2 = \sum_{k\in\mathcal{K}} |c_k|^2 + \sum_{k\in\mathcal{K}} |c_k - a_k|^2.$$

Find two distinct sets $\{a_{m,n}\}_{m,n\in\mathbb{Z}}$ such that $g = \sum_{m,n\in\mathcal{K}} a_{m,n} g_{mT,nF}$, and then use the Lemma to deduce that $TF \leq 1$.

Problem 6 Recovery.

We consider functions of the form

$$x = \sum_{n\in\mathbb{Z}} c_n\, \phi(\,\cdot - nT), \tag{2}$$

where $T > 0$, $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and $\phi : \mathbb{R} \to \mathbb{R}$ are such that (2) converges unconditionally in $L^2(\mathbb{R})$, but otherwise arbitrary. In many cases such functions can be recovered from samples taken at integer multiples of T, even though they do not need to be bandlimited.

a) Fix $T > 0$ and consider the function

$$x(t) = \begin{cases} 1, & 0 \leq t < T \\ 2, & T \leq t < 2T \\ 1, & 2T \leq t < 3T \\ 4, & 3T \leq t < 4T \\ 0, & \text{else} \end{cases}, \quad t \in \mathbb{R}. \tag{3}$$

(i) Sketch x on the interval $[-T, 5T]$, and show that x can be written in the form (2) by specifying suitable $\{c_n\}_{n\in\mathbb{Z}}$ and ϕ.

(ii) By explicitly computing the Fourier transform of x in (3), show that this x is not bandlimited. You may use — without proof — the fact that the set of zeros of a trigonometric polynomial is discrete.

(iii) Note that x can be reconstructed from the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ taken at integer multiples of T (provided that ϕ is known). Seeing that x is not bandlimited, explain why this does not contradict the sampling theorem (Theorem 1.4.1 in the lecture notes).

b) Fix $T > 0$ and let $\phi \in L^2(\mathbb{R})$ be such that $\{\phi(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Furthermore, suppose that ϕ satisfies the following condition:

$$\text{there exists an } \alpha > 0 \text{ s.t. } \Big| \sum_{n\in\mathbb{Z}} \phi(nT)\, e^{-in\theta} \Big| \geq \alpha, \qquad \text{for all } \theta \in [0, 2\pi). \tag{$*$}$$

Now, consider functions x of the form (2) with $\{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(i) Let $\boldsymbol{x} = \{x(nT)\}_{n\in\mathbb{Z}}$. Find elements $\boldsymbol{\phi}^n$ of $\ell^2(\mathbb{Z})$, for $n \in \mathbb{Z}$, such that

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}} c_n \boldsymbol{\phi}^n. \tag{4}$$

(ii) Provide an expression for the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ given $\{\phi(nT)\}_{n\in\mathbb{Z}}$ and the samples $\{x(nT)\}_{n\in\mathbb{Z}}$. You may use — without proof — the fact that the series (4) converges unconditionally, and that $\{x(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Problem 7 Haar wavelets.

Consider the function $\Psi \in L^2(\mathbb{R})$ defined by

$$\psi(x) = \begin{cases} 1, & 0 \leq x < 1/2 \\ -1, & 1/2 \leq x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and set $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$, for $j, k \in \mathbb{Z}$. Prove that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal system. I.e., prove that for all $j, k, m, n \in \mathbb{Z}$, the following holds:

$$\langle \psi_{j,k}, \psi_{m,n} \rangle = \delta_{j,m}\delta_{k,n},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\mathbb{R})$ and $\delta_{\cdot,\cdot}$ is Kronecker's delta.

This orthonormal system can be extended to form an orthonormal basis of $L^2(\mathbb{R})$. You can find more details in the 'Wavelets' notes on the webpage.