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Mathematics of Information

Spring semester 2022

Solutions to Problem Set 3

Problem 1 Unitary operators. ☕

a) $U^H U = \mathrm{Id}$ implies that the N columns of U are orthonormal and hence $\mathrm{Im}(U) = \mathbb{C}^N$. I.e., for each $y \in \mathbb{C}^N$ there exists $x \in \mathbb{C}^N$ such that $y = Ux$. Next, we calculate

$$
\begin{aligned}
& U^H U = \mathrm{Id} && (1)\\
\Rightarrow\quad & U U^H U = U && (2)\\
\Rightarrow\quad & (U U^H - \mathrm{Id}) U = 0 && (3)\\
\Rightarrow\quad & (U U^H - \mathrm{Id}) U x = 0 \quad \forall x \in \mathbb{C}^N. && (4)
\end{aligned}
$$

This implies $(UU^H - \mathrm{Id}) = 0$ since otherwise there would be a $y \in \mathbb{C}^N$ (and in turn a $x \in \mathbb{C}^N$ with $y = Ux$) such that $0 \neq (UU^H - \mathrm{Id})y = (UU^H - \mathrm{Id})Ux$.

b) For every $x \in \ell_2(\mathbb{R})$, we have

$$
\begin{aligned}
(\mathcal{A}^* \mathcal{A} x)_k &= \left(\mathcal{A}^* \left(\sum_{n\in\mathbb{Z}} x_n e_n(\cdot)\right)\right)_k \\
&= \left\langle \sum_{n\in\mathbb{Z}} x_n e_n, e_k \right\rangle \\
&= \sum_{n\in\mathbb{Z}} x_n \langle e_n, e_k \rangle \\
&= x_k, \qquad\qquad \forall k \in \mathbb{Z}.
\end{aligned}
$$

c) We show $\mathcal{A}\mathcal{A}^* \neq \mathrm{Id}$ by finding $y(\cdot) \in L^2(\mathbb{R})$ such that $\mathcal{A}\mathcal{A}^* y \neq y$. We choose

$$
y(t) = t\, e_0(t) = \begin{cases} t & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{else} \end{cases}.
$$

We have $y \in L^2(\mathbb{R})$. Further, $\langle y, e_0 \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} t \,\mathrm{d}t = 0$ and thus also $(\mathcal{A}^* y)_n = 0 \; \forall n \in \mathbb{Z}$ since y is only supported on $(-\frac{1}{2}, \frac{1}{2})$. Therefore, we get $\mathcal{A}\mathcal{A}^* y = \mathcal{A}\, 0 = 0 \neq y$.

d) We have $\mathrm{Im}(\mathcal{A}) \neq L^2(\mathbb{R})$ since, for example, $y(t) \notin \mathrm{Im}(\mathcal{A})$. On the other hand, you can use the argument from subtask (a) to show that for an operator $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_2$ the conditions $\mathcal{B}^*\mathcal{B} = \mathrm{Id}$ and $\mathrm{Im}(\mathcal{B}) = \mathcal{H}_2$ together imply $\mathcal{B}\mathcal{B}^* = \mathrm{Id}$.

Problem 2 Redundancy of a frame.

a) Assume that $\{\mathbf{f}_k\}_{k=1}^N$ is a tight frame for $\mathbb{C}^M$ with frame bound A such that $\|\mathbf{f}_k\| = 1$ for all $1 \le k \le N$. Choose an orthonormal basis $\{\mathbf{e}_k\}_{k=1}^M$ for $\mathbb{C}^M$. Using Parseval's equality and the fact that the $\mathbf{f}_k$, $1 \le k \le N$, are normalized, we obtain

$$1 = \|\mathbf{f}_k\|^2 = \sum_{\ell=1}^{M} |\langle \mathbf{f}_k, \mathbf{e}_\ell \rangle|^2$$

for all $1 \le k \le N$. On the other hand, since $\{\mathbf{f}_k\}_{k=1}^N$ is a tight frame with frame bound A, we have

$$A = A\|\mathbf{e}_\ell\|^2 = \sum_{k=1}^{N} |\langle \mathbf{e}_\ell, \mathbf{f}_k \rangle|^2 \tag{5}$$

for all $1 \le \ell \le M$. Summing (5) for all $1 \le \ell \le M$ yields

$$MA = \sum_{\ell=1}^{M}\sum_{k=1}^{N} |\langle \mathbf{e}_\ell, \mathbf{f}_k \rangle|^2 = \sum_{k=1}^{N}\underbrace{\sum_{\ell=1}^{M} |\langle \mathbf{f}_k, \mathbf{e}_\ell \rangle|^2}_{=1} = N.$$

As a result, we have necessarily $A = N/M$.

b) Assume that $\{\mathbf{f}_k\}_{k=1}^N$ is a frame for $\mathbb{C}^M$ with frame bounds A and B such that $\|\mathbf{f}_k\| = 1$ for all $1 \le k \le N$. As in 1., choose an orthonormal basis $\{\mathbf{e}_k\}_{k=1}^M$ for $\mathbb{C}^M$. Again, using Parseval's equality and the fact that the $\mathbf{f}_k$, $1 \le k \le N$, are normalized, we obtain

$$1 = \|\mathbf{f}_k\|^2 = \sum_{\ell=1}^{M} |\langle \mathbf{f}_k, \mathbf{e}_\ell \rangle|^2$$

for all $1 \le k \le N$. Since $\{\mathbf{f}_k\}_{k=1}^N$ is a frame for $\mathbb{C}^M$ with frame bounds A, B, we have for all $1 \le \ell \le M$ that

$$A = A\|\mathbf{e}_\ell\|^2 \le \sum_{k=1}^{N} |\langle \mathbf{e}_\ell, \mathbf{f}_k \rangle|^2 \le B\|\mathbf{e}_\ell\|^2 = B. \tag{6}$$

Summing (6) for all $1 \le \ell \le M$ then gives

$$AM \le \sum_{\ell=1}^{M}\sum_{n=1}^{N} |\langle \mathbf{e}_\ell, \mathbf{f}_k \rangle|^2 = \sum_{n=1}^{N}\underbrace{\sum_{\ell=1}^{M} |\langle \mathbf{f}_k, \mathbf{e}_\ell \rangle|^2}_{=1} \le BM,$$

which results in $A \le N/M \le B$.

Problem 3 Frame expansion with noise

We have the following:

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{f}-\mathbf{f}_w\|^2\right] &= \mathbb{E}\left[\left\|\frac{1}{A}\sum_{j=1}^{M} w_j \mathbf{g}_j\right\|^2\right] = \mathbb{E}\left[\frac{1}{A^2}\sum_{j=1}^{M}\sum_{\ell=1}^{M} w_j w_\ell \langle \mathbf{g}_j \rangle \mathbf{g}_\ell\right] \\ &= \frac{1}{A^2}\sum_{j=1}^{M}\sum_{\ell=1}^{M} \underbrace{\mathbb{E}\left[w_j w_\ell\right]}_{=\delta_{j,\ell}N_0} \langle \mathbf{g}_j \rangle \mathbf{g}_\ell = \frac{N_0}{A^2}\sum_{j=1}^{M} \underbrace{\|\mathbf{g}_j\|^2}_{=1} \\ &= \frac{N_0 M}{A^2} = \frac{N_0 N}{r}.\end{aligned}$$

For any Hilbert space of dimension N, the MSE is inversely proportional to the redundancy. Therefore, it is an advantage to formulate algorithms involving frames than bases, which have redundancy $r = 1$.

Problem 4 DFT as a signal expansion.

Define the basis functions

$$e_k(n) = \frac{1}{\sqrt{N}} e^{i2\pi \frac{k}{N} n}, \quad k = 0, 1, \ldots, N-1.$$

They form an orthonormal system as shown by

$$\begin{aligned}\langle \mathbf{e}_k, \mathbf{e}_\ell \rangle &= \sum_{n=0}^{N-1} e_k(n)\overline{e_\ell(n)} = \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{i2\pi\frac{k}{N}n} \frac{1}{\sqrt{N}} e^{-i2\pi\frac{\ell}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}(k-\ell)n} \\ &= \begin{cases} \frac{1}{N}\frac{e^{i2\pi(k-\ell)}-1}{e^{i(k-\ell)\frac{2\pi}{N}}-1} = 0, & k \neq \ell \\ \frac{1}{N}\sum_{k=0}^{N-1} 1 = 1, & k = \ell. \end{cases}\end{aligned}$$

We have N functions in $\mathbb{C}^N$ that form an orthonormal system, thus, they form an orthonormal basis. Therefore any signal can be expressed as

$$f(n) = \sum_{k=0}^{N-1} \langle \mathbf{f}, \mathbf{e}_k \rangle \, e_k(n)$$

where

$$\langle \mathbf{f}, \mathbf{e}_k \rangle = \sum_{n=0}^{N-1} f(n)\overline{e_k(n)} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} f(n) e^{-i2\pi\frac{k}{N}n} = \widehat{f}(k).$$

Therefore we see that the inverse of the DFT is given by

$$f(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \widehat{f}(k) e^{i2\pi\frac{k}{N}n}.$$

Problem 5 Weyl-Heisenberg systems.

a) Let $f, g \in L^2(\mathbb{R})$ and $x, y \in \mathbb{R}$ be arbitrary. We have the following reconstruction formula:

$$\begin{aligned} f_{-x,-y} &= \sum_{m,n\in\mathbb{Z}} \langle f_{-x,-y}, \tilde{g}_{mT,nF}\rangle\, g_{mT,nF} \\ &= \sum_{m,n\in\mathbb{Z}} \langle f_{-x,0}, \tilde{g}_{mT,nF+y}\rangle\, g_{mT,nF} \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle\, g_{mT,nF}. \end{aligned}$$

Now we take the inner product of both sides of this equation with $h_{-x,-y}$ to obtain

$$\begin{aligned} \langle f_{-x,-y}, h_{-x,-y}\rangle &= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT,nF}, h_{-x,-y}\rangle \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT,nF+y}, h_{-x,0}\rangle \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle e^{-2\pi i(nF+y)x}\, g_{mT+x,nF+y}, h\rangle \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle. \end{aligned}$$

Finally, note that $\langle f_{-x,-y}, h_{-x,-y}\rangle = \langle f, h\rangle$. Together with the previous equation, this yields the desired identity.

b) Integrating as suggested, we get

$$\begin{aligned} TF\langle f, h\rangle &= \int_0^F \int_0^T \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \qquad (7) \\ &= \sum_{m,n\in\mathbb{Z}} \int_{nF}^{(n+1)F} \int_{mT}^{(m+1)T} \langle f, \tilde{g}_{x,y}\rangle \langle g_{x,y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \langle f, \tilde{g}_{x,y}\rangle \langle g_{x,y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} \langle f, \tilde{g}_{x,y}\rangle \overline{\langle h, g_{x,y}\rangle} \,\mathrm{d}x\,\mathrm{d}y \\ &= \langle f, h\rangle \overline{\langle \tilde{g}, g\rangle}, \end{aligned}$$

where the last step follows by the identity (IR) given in the problem statement. Since f and h were arbitrary (and in particular can be chosen so that $\langle f, h\rangle = 1$), we deduce $\langle g, \tilde{g}\rangle = TF$. To justify the change of order of summation and integration in (7), observe that

$$\begin{aligned} &\sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T |\langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle| \,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{\mathbb{R}^2} |\langle f, \tilde{g}_{x,y}\rangle|\, |\langle g_{x,y}, h\rangle| \,\mathrm{d}x\,\mathrm{d}y \\ &= \langle F_1, F_2\rangle_{L^2(\mathbb{R}^2)} \le \|F_1\|_{L^2(\mathbb{R}^2)} \|F_2\|_{L^2(\mathbb{R}^2)} < \infty, \qquad (8) \end{aligned}$$

where $F_1(x,y) = |\langle f, \tilde{g}_{x,y}\rangle|$ and $F_2(x,y) = |\langle h, g_{x,y}\rangle|$ are $L^2(\mathbb{R}^2)$ functions by the assumption at the beginning of the problem statement.

c) We have the following two expansions:

$$g = \sum_{m,n\in\mathbb{Z}} \langle g, \tilde{g}_{mT,nF}\rangle\, g_{mT,nF} = 1 \cdot g_{0,0} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} 0 \cdot g_{mT,nF}.$$

Now, by the Lemma given in the problem statement, we have

$$\begin{aligned} 1^2 + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} 0^2 &= \sum_{m,n\in\mathbb{Z}} |\langle g, \tilde{g}_{mT,nF}\rangle|^2 + |\langle g, \tilde{g}_{0,0}\rangle - 1|^2 + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} |\langle g, \tilde{g}_{mT,nF}\rangle - 0|^2 \\ &\geq |\langle g, \tilde{g}_{0,0}\rangle|^2 + |\langle g, \tilde{g}_{0,0}\rangle - 1|^2 \geq |\langle g, \tilde{g}\rangle|^2 = (TF)^2, \end{aligned} \tag{9}$$

where the first inequality is obtained by discarding all the terms on the right-hand side of the first equality except for the summands with $(m,n) = (0,0)$. Thus, we have shown that $TF \leq 1$.

Problem 6 Recovery. ☕☕☕

a) (i)

We can write $x = \sum_{n\in\mathbb{Z}} c_n\, \phi(\,\cdot - nT)$, where $\phi = \mathbb{1}_{[0,T)}$, and

$$c_n = \begin{cases} 1, & n \in \{0,2\} \\ 2, & n = 1 \\ 4, & n = 3 \\ 0, & \text{else} \end{cases}.$$

(ii) The Fourier transform of $\phi = \mathbb{1}_{[0,T)}$ is given by

$$\begin{aligned} \hat{\phi}(\omega) &= \int_0^T e^{-2\pi i\omega t}\mathrm{d}t = \left.\frac{e^{-2\pi i t\omega}}{-2\pi i\omega}\right|_{t=0}^{T} = \frac{1 - e^{-2\pi i T\omega}}{2\pi i\omega} \\ &= Te^{-\pi i T\omega}\frac{e^{\pi i T\omega} - e^{-\pi i T\omega}}{2\pi i T\omega} = Te^{-\pi i T\omega}\operatorname{sinc}(T\omega), \quad \omega \in \mathbb{R}, \end{aligned} \tag{10}$$

where $\operatorname{sinc}(\theta) := \frac{\sin(\pi\theta)}{\pi\theta}$, $\theta \in \mathbb{R}$. Therefore, as x is a linear combination of time-shifted versions of ϕ, we have

$$\hat{x}(\omega) = \underbrace{\left(\sum_{n=0}^{3} c_n\, e^{-2\pi i n T\omega}\right)}_{p(\omega)} \cdot \hat{\phi}(\omega), \quad \omega \in \mathbb{R}.$$

Note that $\hat{x}(\omega) = 0$ if and only if at least one of $p(\omega)$ and $\hat{\phi}(\omega)$ is zero. We have $\{\omega \in \mathbb{R} : \hat{\phi}(\omega) = 0\} = \{\frac{n}{T}\}_{n\in\mathbb{Z}\setminus\{0\}}$ from the explicit expression (10). Moreover, as $p(\omega)$ is a non-zero trigonometric polynomial, the set $\{\omega \in \mathbb{R} : p(\omega) = 0\}$ is discrete.

Therefore

$$\{\omega \in \mathbb{R} : \hat{x}(\omega) = 0\} = \{\omega \in \mathbb{R} : \hat{\phi}(\omega) = 0\} \cup \{\omega \in \mathbb{R} : p(\omega) = 0\}$$

is discrete, and hence x is not bandlimited.

(iii) The fact that x can be reconstructed by sampling it at integer multiples of T even though it is not bandlimited does not contradict the sampling theorem as the sampling theorem only states that bandlimitedness is sufficient for reconstruction, but does not claim necessity.

b) (i) Note that

$$x(kT) = \sum_{n\in\mathbb{Z}} c_n \phi(kT - nT), \quad \text{for all } k \in \mathbb{Z},$$

so we simply set $\boldsymbol{\phi}^n = \{\phi((k-n)T)\}_{k\in\mathbb{Z}}$ to obtain

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}} c_n \boldsymbol{\phi}^n. \tag{11}$$

(ii) Note that, as DTFT : $\ell^2(\mathbb{Z}) \to L^2[0, 2\pi)$ is continuous and (11) converges unconditionally, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta} &= \text{DTFT}\{\boldsymbol{x}\}(\theta) = \text{DTFT}\Big\{ \sum_{n\in\mathbb{Z}} c_n \boldsymbol{\phi}^n \Big\}(\theta) \\
&= \sum_{n\in\mathbb{Z}} c_n \, \text{DTFT}\{\boldsymbol{\phi}^n\}(\theta) \\
&= \sum_{n\in\mathbb{Z}} c_n e^{-in\theta} \, \text{DTFT}\{\boldsymbol{\phi}^0\}(\theta) \\
&= \text{DTFT}\{\{c_n\}_{n\in\mathbb{Z}}\}(\theta) \cdot \sum_{k\in\mathbb{Z}} \phi(kT)e^{-ik\theta}, \quad \theta \in [0, 2\pi).
\end{aligned} \tag{12}$$

Now, let $\alpha > 0$ be such that $\left| \sum_{k\in\mathbb{Z}} \phi(kT)\, e^{-ik\theta} \right| \geq \alpha > 0$, for all $\theta \in [0, 2\pi)$, as per the problem assumptions. We can then divide both sides of (12) to obtain

$$\text{DTFT}\{\{c_n\}_{n\in\mathbb{Z}}\}(\theta) = \frac{\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta}}{\sum_{k\in\mathbb{Z}} \phi(kT)e^{-ik\theta}}, \quad \theta \in [0, 2\pi).$$

Finally, inverting the discrete-time Fourier transform, we find

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta}}{\sum_{k\in\mathbb{Z}} \phi(kT)e^{-ik\theta}} \, e^{in\theta} \mathrm{d}\theta, \quad n \in \mathbb{Z}.$$

Problem 7 Haar wavelets.

By definition, we have that $\operatorname{supp} \psi = [0, 1)$ and hence, for all $j, k \in \mathbb{Z}$, it holds that

$$\operatorname{supp} \psi_{j,k} = [2^j k, 2^j (k+1)). \tag{13}$$

Let $(j, k) \neq (j', k')$.

- If $j = j'$ and $k \neq k'$, then by (13), the support of $\psi_{j,k}$ and $\psi_{j',k'}$ are disjoint, i.e.,

$$\operatorname{supp} \psi_{j,k} \cap \operatorname{supp} \psi_{j,k'} = \emptyset,$$

  implying that $\langle \psi_{j,k}, \psi_{j',k'} \rangle = 0$.

- If $j \neq j'$, assume without loss of generality that $j > j'$. Then, $\psi_{j,k}$ is constant on the support of $\psi_{j',k'}$, and since

$$\int_{-\infty}^{+\infty} \psi_{j,k}(x)\mathrm{d}x = \int_{2^j k}^{2^j(k+1)} 2^{-j/2}\psi(2^{-j}x - k)\mathrm{d}x = 2^{j/2}\int_0^1 \psi(x)\mathrm{d}x = 0,$$

  it follows that $\langle \psi_{j,k}, \psi_{j',k'} \rangle = 0$.

This shows that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthogonal system. Moreover, we have for all $j, k \in \mathbb{Z}$,

$$\|\psi_{j,k}\|^2 = \int_{-\infty}^{+\infty} |\psi_{j,k}(x)|^2\mathrm{d}x = \int_{2^j k}^{2^j(k+1)} 2^j|\psi(2^{-j}x - k)|^2\mathrm{d}x = \int_0^1 |\psi(x)|\mathrm{d}x = 1.$$

Therefore, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal system.