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Mathematics of Information

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Problem Set 4

Problem 1 The dual of a Weyl-Heisenberg frame. ☕☕

The Weyl operator $\mathbb{W}_{m,n} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as

$$\mathbb{W}_{m,n} : f(\bullet) \to e^{2\pi i n F \bullet} f(\bullet - mT),$$

where $m, n \in \mathbb{Z}$, and $T, F > 0$ are fixed time- and frequency-shift parameters, respectively. Assume that $\mathcal{G} = \{\mathbb{W}_{k,\ell}\, g\}_{k\in\mathbb{Z},\ell\in\mathbb{Z}}$ with $g \in L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$. Let $\mathbb{S}$ be the corresponding frame operator and $\widetilde{g} = \mathbb{S}^{-1} g$ the canonical dual function. In this exercise we show that the canonical dual frame to $\mathcal{G}$ is given by $\widetilde{\mathcal{G}} = \{\mathbb{W}_{k,\ell}\, \widetilde{g}\}_{k\in\mathbb{Z},\ell\in\mathbb{Z}}$.

a) Show that the adjoint operator of $\mathbb{W}_{m,n}$ is given by

$$(\mathbb{W}_{m,n})^* = e^{-2\pi i n m T F}\, \mathbb{W}_{-m,-n}\ .$$

b) Show that for $m, n, k, \ell \in \mathbb{Z}$

$$\mathbb{W}_{m,n}\mathbb{W}_{k,\ell} = e^{-2\pi i \ell m T F}\mathbb{W}_{m+k,n+\ell}\ .$$

c) Show that for all $m, n \in \mathbb{Z}$

$$\mathbb{W}_{m,n}\, \mathbb{S} = \mathbb{S}\, \mathbb{W}_{m,n}\ .$$

d) Show that for all $m, n \in \mathbb{Z}$

$$\mathbb{W}_{m,n}\, \mathbb{S}^{-1} = \mathbb{S}^{-1}\, \mathbb{W}_{m,n}\ .$$

e) Show that the canonical dual frame to $\mathcal{G}$ is given by $\widetilde{\mathcal{G}} = \{\mathbb{W}_{k,\ell}\, \widetilde{g}\}_{k\in\mathbb{Z},\ell\in\mathbb{Z}}$.

Problem 2 Comparison of norms of matrices. ☕

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ be a nonzero matrix. Prove that

$$\frac{\|\mathbf{A}\|_2}{\sqrt{\operatorname{rank}(\mathbf{A})}} \leq |||\mathbf{A}|||_2 \leq \|\mathbf{A}\|_2.$$

Problem 3 Coherence of a matrix. ☕

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ and $\mathbf{B} \in \mathbb{C}^{m \times m}$ be unitary matrices. We recall that the coherence of a matrix $\mathbf{M} \in \mathbb{C}^{m \times n}$ with columns $\{\mathbf{m}_1, \ldots, \mathbf{m}_n\}$ which are $\|\cdot\|_2$-normalized to 1, is defined as

$$\mu(\mathbf{M}) \coloneqq \max_{i \neq j} \left| \mathbf{m}_i^H \mathbf{m}_j \right|.$$

a) Let $\mathbf{U} = \mathbf{A}^H \mathbf{B}$. Prove that $\mathbf{U}$ is unitary.

b) Prove that $\mu([\mathbf{A} \;\; \mathbf{B}]) = \mu([\mathbf{I} \;\; \mathbf{U}]) = \max_{i,j} |\mathbf{U}_{i,j}|$, where $\mathbf{I}$ is the identity matrix.

c) Prove that $\frac{1}{m} \leq \mu^2([\mathbf{I} \;\; \mathbf{U}]) \leq 1$ and provide examples for which the inequalities are tight.

Problem 4 Operator norm of orthogonal projections. ☕☕

Let $\mathbf{U} \in \mathbb{C}^{m \times m}$ be a unitary matrix and $\mathcal{P}, \mathcal{Q} \subseteq \{1, \ldots, m\}$ be index sets. We further define, as in the lecture, the orthogonal projection

$$\mathbf{P}_{\mathcal{Q}}(\mathbf{U}) \coloneqq \mathbf{U}\mathbf{D}_{\mathcal{Q}}\mathbf{U}^H$$

on the subspace $\mathcal{W}^{\mathbf{U},\mathcal{Q}}$, where, given an index set $\mathcal{I} \subseteq \{1, \ldots, m\}$, $\mathbf{D}_{\mathcal{I}} \in \mathbb{C}^{m \times m}$ denotes the diagonal matrix with

$$(\mathbf{D}_{\mathcal{I}})_{i,i} = \begin{cases} 1, & \text{if } i \in \mathcal{I}, \\ 0, & \text{otherwise.} \end{cases}$$

a) Prove that $|||\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{D}_{\mathcal{P}}|||_2 = |||\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})|||_2$.

b) Prove that $|||\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})|||_2 = \max_{\mathbf{x} \in \mathcal{W}^{\mathbf{U},\mathcal{Q}} \setminus \{0\}} \frac{\|\mathbf{x}_{\mathcal{P}}\|_2}{\|\mathbf{x}\|_2}$.

Problem 5 Uncertainty Principle (Exam 2021, Problem 3). ☕☕☕

In this problem, we derive a continuous-time version of an uncertainty relation presented in the lecture: the so-called Donoho-Stark uncertainty principle. Specifically, we consider a complex-valued signal $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of unit L^2-norm, i.e., $\|f\|_2 = 1$ and write $\hat{f}$ for its Fourier transform. We further introduce the time-limiting operator $P_{\mathcal{T}}$ and the frequency-limiting operator $P_{\mathcal{W}}$, defined as

$$(P_{\mathcal{T}}f)(t) = \mathbb{1}_{\mathcal{T}}(t)f(t) \quad \text{and} \quad (P_{\mathcal{W}}f)(t) = \int_{\mathcal{W}} e^{2\pi i w t}\hat{f}(w)dw,$$

where $\mathcal{T}$ and $\mathcal{W}$ are bounded subsets of $\mathbb{R}$, and $\mathbb{1}_{\mathcal{T}}$ is the indicator of $\mathcal{T}$, i.e.,

$$\mathbb{1}_{\mathcal{T}}(t) = \begin{cases} 1, & \text{if } t \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Further, the signal f considered is $\varepsilon_{\mathcal{T}}$-concentrated to $\mathcal{T}$ and $\varepsilon_{\mathcal{W}}$-concentrated to $\mathcal{W}$ according to

$$\|f - P_{\mathcal{T}}f\|_2 \leq \varepsilon_{\mathcal{T}} \quad \text{and} \quad \|f - P_{\mathcal{W}}f\|_2 \leq \varepsilon_{\mathcal{W}}.$$

For the operator P, we write $\|P\|_{2\to 2} \coloneqq \sup_{\|g\|_2=1} \|Pg\|_2$ for its operator norm.

a) Show that

$$\|f - P_{\mathcal{W}} P_{\mathcal{T}} f\|_2 \leq \varepsilon_{\mathcal{T}} + \varepsilon_{\mathcal{W}}.$$

Hint: First prove and then use that $\|P_{\mathcal{W}}\|_{2\to 2} = 1$.

b) Show that

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2} \geq 1 - \varepsilon_{\mathcal{T}} - \varepsilon_{\mathcal{W}}.$$

Hint: You can use, without proof, the reverse triangle inequality, namely that, for all $g, h \in L^2(\mathbb{R})$, one has $\|g - h\|_2 \geq \|g\|_2 - \|h\|_2$.

c) Show that, for all $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|g\|_2 = 1$, we have

$$(P_{\mathcal{W}} P_{\mathcal{T}} g)(s) = \int_{-\infty}^{\infty} q(s,t) g(t) dt,$$

for some $q(s,t)$ to be expressed explicitly, and use this relation to prove that

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2}^2 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 dt\, ds.$$

Hint: Apply Fubini's theorem.

d) Prove the following identity

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 dt\, ds = |\mathcal{W}||\mathcal{T}|.$$

Hint: First express the function q in terms of the inverse Fourier transform of an indicator function and then use the Plancherel identity.

e) Combine the results established in the previous subproblems to prove that

$$|\mathcal{W}||\mathcal{T}| \geq (1 - (\varepsilon_{\mathcal{T}} + \varepsilon_{\mathcal{W}}))^2.$$