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Mathematics of Information

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Solutions to Problem Set 4

Problem 1 The dual of a Weyl-Heisenberg frame.

a) For all $h, f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\langle \mathbb{W}_{m,n} h, f \rangle &= \int_{-\infty}^{\infty} e^{2\pi i n F t} h(t - mT) \overline{f(t)}\, dt \\
&= \int_{-\infty}^{\infty} e^{2\pi i n F(t' + mT)} h(t') \overline{f(t' + mT)}\, dt' \\
&= \int_{-\infty}^{\infty} h(t') \overline{e^{-2\pi i n m T F} e^{2\pi i (-n) F t'} f(t' - (-m)T)}\, dt' \\
&= \int_{-\infty}^{\infty} h(t') \overline{e^{-2\pi i n m T F} (\mathbb{W}_{-m,-n} f)(t')}\, dt' \\
&= \left\langle h, e^{-2\pi i n m T F}\, \mathbb{W}_{-m,-n} f \right\rangle \\
&= \langle h, (\mathbb{W}_{m,n})^* f \rangle,
\end{aligned} \tag{1}$$

which establishes that $(\mathbb{W}_{m,n})^* = e^{-2\pi i n m T F}\, \mathbb{W}_{-m,-n}$.

b) For all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\mathbb{W}_{m,n} \mathbb{W}_{k,\ell}\, f &= \mathbb{W}_{m,n} \left(e^{2\pi i \ell F \bullet} f(\bullet - kT) \right) \\
&= e^{2\pi i n F \bullet} e^{2\pi i \ell F(\bullet - mT)} f(\bullet - mT - kT) \\
&= e^{-2\pi i (lF)(mT)} e^{2\pi i (n+\ell) F \bullet} f(\bullet - (m+k)T) \\
&= e^{-2\pi i \ell m T F} \mathbb{W}_{m+k, n+\ell}\, f
\end{aligned} \tag{2}$$

c) For all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\mathbb{S}(\mathbb{W}_{m,n} f) &= \sum_{k,\ell \in \mathbb{Z}^2} \langle \mathbb{W}_{m,n}\, f, (\mathbb{W}_{k,\ell}\, g) \rangle\, (\mathbb{W}_{k,\ell}\, g) && (3)\\
&= \sum_{k,\ell \in \mathbb{Z}^2} \left\langle f, e^{-2\pi i n m T F} \mathbb{W}_{-m,-n}\, \mathbb{W}_{k,\ell}\, g \right\rangle (\mathbb{W}_{k,\ell}\, g) && (4)\\
&= \sum_{k,\ell \in \mathbb{Z}^2} \left\langle f, e^{-2\pi i n m T F} e^{2\pi i \ell m T F} \mathbb{W}_{k-m,\ell-n}\, g \right\rangle (\mathbb{W}_{k,\ell}\, g) && (5)\\
&= \sum_{k,\ell \in \mathbb{Z}^2} \langle f, \mathbb{W}_{k-m,\ell-n}\, g \rangle \left(e^{-2\pi i (\ell - n) m T F} \mathbb{W}_{k,\ell}\, g \right) && (6)\\
&= \sum_{k',\ell' \in \mathbb{Z}^2} \langle f, \mathbb{W}_{k',\ell'}\, g \rangle \left(e^{-2\pi i \ell' m T F} \mathbb{W}_{k'+m,\ell'+n}\, g \right) && (7)\\
&= \sum_{k',\ell' \in \mathbb{Z}^2} \langle f, \mathbb{W}_{k',\ell'}\, g \rangle\, (\mathbb{W}_{m,n} \mathbb{W}_{k',\ell'}\, g) && (8)\\
&= \mathbb{W}_{m,n} \sum_{k',\ell' \in \mathbb{Z}^2} \langle f, \mathbb{W}_{k',\ell'}\, g \rangle\, (\mathbb{W}_{k',\ell'}\, g) && (9)\\
&= \mathbb{W}_{m,n} \mathbb{S}\, f, && (10)
\end{aligned}$$

where in (3) we use the definition of $\mathbb{S}$, in (4) we use (1), in (5) we use (2), and in (6) we use that the inner product is antilinear in the second argument. In (7) we did a change of variable, in (8) we use (2) again, and (9) uses that $\mathbb{W}_{m,n}$ is a linear operator, which is easy to verify.

d) We multiply $\mathbb{S}^{-1}$ from the right to both sides of the equality $\mathbb{W}_{m,n}\, \mathbb{S} = \mathbb{S}\, \mathbb{W}_{m,n}$ from the previous subtask and obtain $\mathbb{W}_{m,n} = \mathbb{S}\, \mathbb{W}_{m,n} \mathbb{S}^{-1}$. Next we multiply $\mathbb{S}^{-1}$ from the left and get $\mathbb{W}_{m,n}\, \mathbb{S}^{-1} = \mathbb{S}^{-1}\, \mathbb{W}_{m,n}$ as desired.

e) The canonical dual frame to a frame $\{g'_k\}_{k \in \mathcal{K}}$ is defined as the set $\{\mathbb{S}^{-1}(g'_k)\}_{k \in \mathcal{K}}$. Hence, the canonical dual to $\mathcal{G}$ is given by

$$\widetilde{\mathcal{G}} = \{\mathbb{S}^{-1}(\mathbb{W}_{k,\ell}\, g)\}_{k \in \mathbb{Z},\, \ell \in \mathbb{Z}} = \{\mathbb{W}_{k,\ell}\, \mathbb{S}^{-1}\, g\}_{k \in \mathbb{Z},\, \ell \in \mathbb{Z}} = \{\mathbb{W}_{k,\ell}\, \widetilde{g}\}_{k \in \mathbb{Z},\, \ell \in \mathbb{Z}},$$

where we used the previous subtask.

Problem 2 Comparison of norms of matrices.

Since $\mathbf{A} \in \mathbb{C}^{m \times n}$ is nonzero, the rank of $\mathbf{A}$ is a well defined positive integer r. Moreover, we can write $\sigma_1 \geq \cdots \geq \sigma_r > 0$ for the nonzero singular values of $\mathbf{A}$, and its singular value decomposition

$$\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^H,$$

where $\Sigma \in \mathbb{C}^{m \times n}$ is a diagonal matrix with diagonal elements $\{\sigma_1, \ldots, \sigma_r, 0, \ldots, 0\}$, and $\mathbf{U} \in \mathbb{C}^{m \times m}$ and $\mathbf{V} \in \mathbb{C}^{n \times n}$ are unitary matrices. In particular, we have

$$|||\mathbf{A}|||_2 = \sup_{\|\mathbf{x}\|_2 = 1} \left\| \mathbf{U} \Sigma \mathbf{V}^H \mathbf{x} \right\|_2 = \sup_{\|\mathbf{x}\|_2 = 1} \left\| \Sigma \mathbf{V}^H \mathbf{x} \right\|_2 = \sup_{\|\mathbf{y}\|_2 = 1} \|\Sigma \mathbf{y}\|_2 = \sigma_1,$$

where we successively used that $\mathbf{U}$ and $\mathbf{V}$ are unitary, and

$$\|\mathbf{A}\|_2 = \sqrt{\mathrm{Tr}\left[\mathbf{A}^H\mathbf{A}\right]} = \sqrt{\mathrm{Tr}\left[\mathbf{V}\Sigma^H\mathbf{U}^H\mathbf{U}\Sigma\mathbf{V}^H\right]} = \sqrt{\mathrm{Tr}\left[\Sigma^H\Sigma\right]} = \sqrt{\sum_{i=1}^{r}\sigma_i^2}.$$

We have proven

$$|||\mathbf{A}|||_2 = \sigma_1 \quad \text{and} \quad \|\mathbf{A}\|_2 = \sqrt{\sum_{i=1}^{r}\sigma_i^2}.$$

From the inequalities

$$\sqrt{\sum_{i=1}^{r}\sigma_i^2} \leq \sqrt{\sum_{i=1}^{r}\sigma_1^2} = \sqrt{r}\sigma_1 \quad \text{and} \quad \sigma_1^2 \leq \sum_{i=1}^{r}\sigma_i^2,$$

one easily obtains the desired result

$$\frac{\|\mathbf{A}\|_2}{\sqrt{\mathrm{rank}(\mathbf{A})}} \leq |||\mathbf{A}|||_2 \leq \|\mathbf{A}\|_2.$$

Problem 3 Coherence of a matrix.

a) We compute

$$\mathbf{U}^H\mathbf{U} = \mathbf{B}^H\mathbf{A}\mathbf{A}^H\mathbf{B} = \mathbf{B}^H\mathbf{B} = \mathbf{I},$$

where we successively used that $\mathbf{A}$ and $\mathbf{B}$ are unitary. A similar computation gives $\mathbf{U}\mathbf{U}^H = \mathbf{I}$, from which we conclude that $\mathbf{U}$ is unitary.

b) We apply the definition of the coherence of a matrix to obtain

$$\mu([\mathbf{A} \;\; \mathbf{B}]) = \max\left\{\max_{i\neq j}\left|\mathbf{a}_i^H\mathbf{a}_j\right|, \max_{i\neq j}\left|\mathbf{b}_i^H\mathbf{b}_j\right|, \max_{i,j}\left|\mathbf{a}_i^H\mathbf{b}_j\right|\right\}.$$

Since $\mathbf{A}$ and $\mathbf{B}$ are unitary, we have

$$\max_{i\neq j}\left|\mathbf{a}_i^H\mathbf{a}_j\right| = 0 \quad \text{and} \quad \max_{i\neq j}\left|\mathbf{b}_i^H\mathbf{b}_j\right| = 0,$$

which implies the following equation

$$\mu([\mathbf{A} \;\; \mathbf{B}]) = \max_{i,j}\left|\mathbf{a}_i^H\mathbf{b}_j\right|.$$

Applying the same reasoning with $\mathbf{I}$ in place of $\mathbf{A}$ and $\mathbf{U}$ in place of $\mathbf{B}$ yields

$$\mu([\mathbf{I} \;\; \mathbf{U}]) = \max_{i,j}\left|\mathbf{e}_i^H\mathbf{U}_j\right| = \max_{i,j}|\mathbf{U}_{i,j}|,$$

where $\mathbf{e}_i$ is the vector with a 1 at the i-th position and zeros elsewhere. Observing that $\mathbf{U}_{i,j} = \mathbf{a}_i^H\mathbf{b}_j$, by definition of $\mathbf{U}$, the desired result follows:

$$\mu([\mathbf{A} \;\; \mathbf{B}]) = \mu([\mathbf{I} \;\; \mathbf{U}]) = \max_{i,j}|\mathbf{U}_{i,j}|.$$

c) We derive the upper bound by applying Cauchy-Schwarz inequality:

$$\mu([\mathbf{I} \;\; \mathbf{U}]) = \max_{i,j} \left| \mathbf{e}_i^H \mathbf{U}_j \right| \leq \max_{i,j} \|\mathbf{e}_i\|_2 \, \|\mathbf{U}_j\|_2 = 1.$$

From the equality case in the Cauchy-Schwarz inequality, we deduce that choosing $\mathbf{U} = \mathbf{I}$ yields $\mu([\mathbf{I} \;\; \mathbf{I}]) = 1$.

For the lower bound, we proceed as follows:

$$\mu^2([\mathbf{I} \;\; \mathbf{U}]) = \max_{i,j} |\mathbf{U}_{i,j}|^2 \geq \frac{1}{m} \sum_{j=1}^{m} |\mathbf{U}_{1,j}|^2 = \frac{1}{m} \left\| \mathbf{U}_1^H \right\|_2^2 = \frac{1}{m}.$$

By the previous derivation, one observes that the equality case corresponds to the matrices which satisfy $|\mathbf{U}_{i,j}| = |\mathbf{U}_{i,k}|$ for all i, j and k. For instance, choosing $\mathbf{U}$ to be the DFT matrix, i.e.,

$$\mathbf{U} = \mathbf{F} := \left(\frac{e^{-2\pi i k \ell / m}}{\sqrt{m}} \right)_{k,\ell},$$

one verifies that

$$\mu^2([\mathbf{I} \;\; \mathbf{F}]) = \max_{k,\ell} |\mathbf{F}_{k,\ell}|^2 = \frac{1}{m}.$$

Problem 4 Operator norm of orthogonal projections.

a) First, we observe that the operators $\mathbf{P}_{\mathcal{Q}}(\mathbf{U})$ and $\mathbf{D}_{\mathcal{P}}$ are self-adjoint. Indeed, $\mathbf{D}_{\mathcal{I}}$ being diagonal with real valued components for every $\mathcal{I}$, it is clearly self-adjoint. This immediatly implies that $\mathbf{D}_{\mathcal{P}}$ is self-adjoint and $\mathbf{P}_{\mathcal{Q}}(\mathbf{U})$ is self-adjoint according to

$$\mathbf{P}_{\mathcal{Q}}(\mathbf{U})^H = \left(U^H\right)^H \mathbf{D}_{\mathcal{Q}}^H U^H = U \mathbf{D}_{\mathcal{Q}} U^H = \mathbf{P}_{\mathcal{Q}}(\mathbf{U}).$$

We further recall that, for a general matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$, we have

$$|||\mathbf{A}|||_2 = \sigma_1 = \left|\left|\left| \mathbf{A}^H \right|\right|\right|_2,$$

where σ_1 is the largest singular value of $\mathbf{A}$. Combining this observation with the self-adjointness of $\mathbf{P}_{\mathcal{Q}}(\mathbf{U})$ and $\mathbf{D}_{\mathcal{P}}$, one obtains the desired result

$$|||\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{D}_{\mathcal{P}}|||_2 = \left|\left|\left| (\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{D}_{\mathcal{P}})^H \right|\right|\right|_2 = \left|\left|\left| \mathbf{D}_{\mathcal{P}}^H \mathbf{P}_{\mathcal{Q}}(\mathbf{U})^H \right|\right|\right|_2 = |||\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})|||_2.$$

b) We first observe that the righ-hand-side can be reformulated as

$$\max_{\mathbf{x} \in \mathcal{W}^{\mathbf{U},\mathcal{Q}} \setminus \{0\}} \frac{\|\mathbf{x}_{\mathcal{P}}\|_2}{\|\mathbf{x}\|_2} = \max_{\mathbf{x} | \mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x} \neq 0} \frac{\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}{\|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}$$

We will proceed in two steps. First, we provide the following upper bound

$$\begin{aligned}\max_{\mathbf{x}|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\neq 0}\frac{\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}{\|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2} &= \max_{\mathbf{x}|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\neq 0}\left\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\frac{\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}}{\|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}\right\|_2\\ &\leq \max_{\|\mathbf{y}\|_2=1}\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{y}\|_2\\ &= |||\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})|||_2.\end{aligned}$$

We now turn to the lower bound:

$$\begin{aligned}\max_{\mathbf{x}|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\neq 0}\frac{\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}{\|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2} &\geq \max_{\|\mathbf{x}\|_2=1,\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\neq 0}\frac{\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}{\|\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2}\\ &\geq \max_{\|\mathbf{x}\|_2=1,\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\neq 0}\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2\\ &= \max_{\|\mathbf{x}\|_2=1}\|\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})\mathbf{x}\|_2\\ &= |||\mathbf{D}_{\mathcal{P}}\mathbf{P}_{\mathcal{Q}}(\mathbf{U})|||_2.\end{aligned}$$

Combining all the inequalities yields the desired result.

Problem 5 Uncertainty Principle (Exam 2021, Problem 3).

a) First, recall the definition of the operator norm as $\|P_{\mathcal{W}}\|_{2\to 2} := \sup_{\|g\|_2=1}\|P_{\mathcal{W}}g\|_2$, which implies that, for all $h \in L^2(\mathbb{R})$ with $h \neq 0$,

$$\|P_{\mathcal{W}}h\|_2 = \left\|P_{\mathcal{W}}\frac{h}{\|h\|_2}\right\|_2\|h\|_2 \leq \sup_{\|g\|_2=1}\|P_{\mathcal{W}}\,g\|_2\|h\|_2 = \|P_{\mathcal{W}}\|_{2\to 2}\|h\|_2. \tag{11}$$

We follow the Hint and observe that for all $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\|P_{\mathcal{W}}g\|_2^2 \overset{\text{Plancherel}}{=} \left\|\widehat{P_{\mathcal{W}}g}\right\|_2^2 = \int_{\mathcal{W}}|\hat{g}(w)|^2\,dw \leq \int_{-\infty}^{\infty}|\hat{g}(w)|^2\,dw \overset{\text{Plancherel}}{=} \|g\|_2^2,$$

and that if moreover g has its Fourier transform $\hat{g}$ supported on $\mathcal{W}$, then $P_{\mathcal{W}}g = g$. We have therefore proven that

$$1 \leq \|P_{\mathcal{W}}\|_{2\to 2} \leq 1,$$

and hence

$$\|P_{\mathcal{W}}\|_{2\to 2} = 1. \tag{12}$$

Next, we note that

$$\begin{aligned}\|f - P_{\mathcal{W}}P_{\mathcal{T}}f\|_2 &\overset{\text{triang. ineq.}}{\leq} \|f - P_{\mathcal{W}}f\|_2 + \|P_{\mathcal{W}}f - P_{\mathcal{W}}P_{\mathcal{T}}f\|_2\\ &\overset{(11)}{\leq} \|f - P_{\mathcal{W}}f\|_2 + \|P_{\mathcal{W}}\|_{2\to 2}\|f - P_{\mathcal{T}}f\|_2\\ &\overset{(12)}{=} \|f - P_{\mathcal{W}}f\|_2 + \|f - P_{\mathcal{T}}f\|_2 \leq \varepsilon_{\mathcal{W}} + \varepsilon_{\mathcal{T}},\end{aligned} \tag{13}$$

where the last inequality holds as f is $\varepsilon_{\mathcal{T}}$-concentrated to $\mathcal{T}$ and simultaneously $\varepsilon_{\mathcal{W}}$-concentrated to $\mathcal{W}$.

b) As it has been assumed in the problem statement that $\|f\|_2 = 1$, applying the reverse triangle inequality yields the desired result according to

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2} \overset{(11)}{\geq} \|P_{\mathcal{W}} P_{\mathcal{T}} f\|_2 = \|f - (f - P_{\mathcal{W}} P_{\mathcal{T}} f)\|_2 \overset{\text{RTI}}{\geq} \|f\|_2 - \|f - P_{\mathcal{W}} P_{\mathcal{T}} f\|_2 \overset{(13)}{\geq} 1 - \varepsilon_{\mathcal{T}} - \varepsilon_{\mathcal{W}}.$$

c) Plugging in the definition of $P_{\mathcal{W}} P_{\mathcal{T}} g$, we obtain

$$\begin{aligned}(P_{\mathcal{W}} P_{\mathcal{T}} g)(s) &= \int_{\mathcal{W}} e^{2\pi i w s} \left(\widehat{\mathbb{1}_{\mathcal{T}}\, g}\right)(w)\, dw \\ &= \int_{\mathcal{W}} \int_{-\infty}^{\infty} e^{2\pi i w(s-t)}\, \mathbb{1}_{\mathcal{T}}(t)\, g(t)\, dt\, dw && (14) \\ &\overset{(*)}{=} \int_{-\infty}^{\infty} \left\{ \int_{\mathcal{W}} e^{2\pi i w(s-t)}\, dw\, \mathbb{1}_{\mathcal{T}}(t) \right\} g(t)\, dt \\ &= \int_{-\infty}^{\infty} q(s,t) g(t)\, dt, && (15)\end{aligned}$$

where $(*)$ follows from Fubini's theorem, and we set

$$q(s,t) = \int_{\mathcal{W}} e^{2\pi i w(s-t)} dw\, \mathbb{1}_{\mathcal{T}}(t).$$

The condition for the application of Fubini's theorem, namely absolute integrability in (14), is satisfied as $\mathcal{T}$ and $\mathcal{W}$ are bounded sets. Now, fixing $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\|g\|_2 = 1$ and using (15), we obtain

$$\begin{aligned}\|P_{\mathcal{W}} P_{\mathcal{T}} g\|_2^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} q(s,t) g(t)\, dt \right|^2 ds \\ &\overset{\text{C.S.}}{\leq} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |q(s,t)|^2\, dt \underbrace{\int_{-\infty}^{\infty} |g(u)|^2\, du}_{=1} \right\} ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2\, dt\, ds, && (16)\end{aligned}$$

where C.S. stands for 'Cauchy-Schwarz inequality'. As the right hand side of (16) does not depend on g, we can conclude, by taking the supremum over all g satisfying $\|g\|_2 = 1$, that, as desired,

$$\|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2}^2 = \sup_{\|g\|_2 = 1} \|P_{\mathcal{W}} P_{\mathcal{T}} g\|_2^2 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 dt\, ds.$$

d) We observe that

$$q(s+t,t) = \int_{-\infty}^{\infty} e^{2\pi i w s} \mathbb{1}_{\mathcal{W}}(w)\, dw \cdot \mathbb{1}_{\mathcal{T}}(t) = \mathcal{F}^{-1}\{\mathbb{1}_{\mathcal{W}}\}(s) \cdot \mathbb{1}_{\mathcal{T}}(t),$$

where $\mathcal{F}^{-1}\{\mathbb{1}_{\mathcal{W}}\}(s)$ is the inverse Fourier transform of the indicator function $\mathbb{1}_{\mathcal{W}}$ evaluated at s. This yields

$$
\begin{aligned}
\int_{-\infty}^{\infty} |q(s,t)|^2 \, ds &= \int_{-\infty}^{\infty} |q(s+t,t)|^2 \, ds \\
&= \int_{-\infty}^{\infty} \left|\mathcal{F}^{-1}\{\mathbb{1}_{\mathcal{W}}\}(s)\right|^2 \, ds \cdot \mathbb{1}_{\mathcal{T}}(t) \\
&\overset{\text{Pl.}}{=} \int_{-\infty}^{\infty} \left|\mathcal{F}\mathcal{F}^{-1}\{\mathbb{1}_{\mathcal{W}}\}(w)\right|^2 \, dw \cdot \mathbb{1}_{\mathcal{T}}(t) \\
&= \int_{-\infty}^{\infty} \mathbb{1}_{\mathcal{W}}(w) \, dw \cdot \mathbb{1}_{\mathcal{T}}(t), \qquad (17)
\end{aligned}
$$

where we used the Plancherel identity, abbreviated as 'Pl.'. Upon integration over t, (17) results in

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 ds\, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{1}_{\mathcal{W}}(w) \mathbb{1}_{\mathcal{T}}(t) \, dw\, dt = |\mathcal{W}||\mathcal{T}|. \qquad (18)
$$

As $\mathcal{T}$ and $\mathcal{W}$ are bounded sets by assumption, the right hand side of (18) is finite and we can hence apply Fubini's theorem to conclude, as desired, that

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 dt\, ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 ds\, dt = |\mathcal{W}||\mathcal{T}|.
$$

e) We combine the results established in the previous subproblems according to

$$
|\mathcal{W}||\mathcal{T}| \overset{(d)}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(s,t)|^2 dt\, ds \overset{(c)}{\geq} \|P_{\mathcal{W}} P_{\mathcal{T}}\|_{2\to 2}^2 \overset{(b)}{\geq} \left(1 - (\varepsilon_{\mathcal{T}} + \varepsilon_{\mathcal{W}})\right)^2 .
$$