ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Chair for Mathematical Information Science

Prof. Dr. H. Bölcskei
Sternwartstrasse 7
CH-8092 Zürich

Mathematics of Information

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Problem Set 5

Problem 1 "ℓ_0-norm". ☕

In the lecture, we defined $\|\mathbf{x}\|_0$ to be the number of nonzero entries in the vector $\mathbf{x} \in \mathbb{C}^N$. Show that $\|\cdot\|_0$ is actually not a norm on $\mathbb{C}^N$, despite the fact that it is often referred to as "ℓ_0-norm".

Problem 2 Gram matrix. ☕

Let $\{\mathbf{a}_k\}_{k=1}^N$ be a set of vectors of $\mathbb{C}^M$. Show that the Gram matrix $\{\langle \mathbf{a}_\ell, \mathbf{a}_k \rangle\}_{k,\ell=1}^N$ is a Hermitian positive-semidefinite matrix, and that it is positive-definite whenever $\{\mathbf{a}_k\}_{k=1}^N$ forms a linearly independent set of vectors.

Problem 3 Summer Exam 2021 ☕☕☕

Notation: For a vector $u \in \mathbb{C}^N$, a matrix $B \in \mathbb{C}^{m\times N}$, and a set $S \subset \{1,\ldots,N\}$, we define $u_S \in \mathbb{C}^{|S|}$ to be the vector obtained from u by keeping only the entries indexed by S, and similarly, we define $B_S \in \mathbb{C}^{m\times|S|}$ to be the matrix obtained from B by keeping only the columns indexed by S. Further, $S^c := \{1,\ldots,N\}\backslash S$ denotes the complement of the set S in $\{1,\ldots,N\}$. B^H stands for the conjugate transpose of the matrix B and $\mathcal{N}(B)$ refers to the null space of B (i.e., $\mathcal{N}(B) = \{v \in \mathbb{C}^N \mid Bv = 0\}$).

In compressed sensing, we are given a measurement vector $y \in \mathbb{C}^m$ obtained according to $y = Dx$, where $x \in \mathbb{C}^N$, $x \neq 0$, is the unknown (sparse) vector to be recovered and $D \in \mathbb{C}^{m\times N}$ is the so-called measurement matrix. In class, we studied two algorithms for recovering x from the observation y, namely

$$\arg\min_{\widehat{x}} \|\widehat{x}\|_0 \quad \text{subject to } D\widehat{x} = y \tag{P0}$$

and

$$\arg\min_{\widehat{x}} \|\widehat{x}\|_1 \quad \text{subject to } D\widehat{x} = y. \tag{P1}$$

a) For this subproblem, we fix

$$D := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -\frac{4}{5} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad x := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and hence

$$y = Dx = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

i) Compute $\mathcal{N}(D)$.

ii) Is the condition $\|x\|_0 < \frac{\text{spark}(D)}{2}$ satisfied?

iii) Is the condition

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right),$$

satisfied? Here, $\mu(D)$ denotes the coherence of D.

iv) Specify the solution set for the linear system of equations $y = D\widehat{x}$, i.e., determine

$$\mathcal{X} := \{\widehat{x} \mid y = D\widehat{x}\}.$$

v) Is x uniquely recovered through (P0)?

vi) Is x uniquely recovered through (P1)?

b) In the following subproblem, we establish sufficient conditions for recovery through (P1). Specifically, these conditions are in terms of the sign pattern of the vector $x \in \mathbb{C}^N$ to be recovered. We define the support set of x as $S = \{i \mid x_i \neq 0\} \subset \{1, \dots, N\}$, and let the complex sign-function $\text{sgn}(\bullet) : \mathbb{C}^N \to \mathbb{C}^N$ be given by

$$(\text{sgn}(x))_k = \begin{cases} x_k/|x_k|, & \text{if } x_k \neq 0 \\ 0, & \text{else} \end{cases}.$$

Throughout we assume that $x \neq 0$.

i) Show that x can be recovered through (P1) if it satisfies the following sufficient condition (C1):

$$\left|\sum_{j \in S} v_j \overline{(\text{sgn}(x))_j}\right| < \sum_{k \in S^c} |v_k|, \qquad \text{for all } v \in \mathcal{N}(D) \setminus \{0\}. \tag{C1}$$

Hint: First show that $\forall u, v \in \mathbb{C}^N : |\langle u, v\rangle| \leq \|u\|_1 \|v\|_\infty$. (1 point)

ii) Show that x can be recovered through (P1) if it satisfies the following sufficient condition (C2):

$$\begin{gathered} \mathcal{N}(D_S) = \{0\} \qquad \text{and there exists } h \in \mathbb{C}^m \text{ s.t.} \\ (D^H h)_j = \text{sgn}(x)_j,\ \forall j \in S, \qquad |(D^H h)_k| < 1,\ \forall k \in S^c. \end{gathered} \tag{C2}$$

Hint: Show that (C2) implies (C1).

iii) Show that x can be recovered through (P1) if it fulfills the following sufficient condition (C3):

$$\mathcal{N}(D_S) = \{0\} \qquad \text{and} \qquad \left|\left\langle D_S^\dagger d_k, \text{sgn}(x_S)\right\rangle\right| < 1,\ \forall k \in S^c, \tag{C3}$$

where d_j denotes the j-th column of D and $D_S^\dagger := ((D_S)^H D_S)^{-1}(D_S)^H$ is the pseudo-inverse of D_S.
<u>Hint</u>: Show that (C3) implies (C2) by proving that (C2) is satisfied by the vector $h = (D_S^\dagger)^H \operatorname{sgn}(x_S)$. You may use without proof that the inverse of a symmetric matrix is symmetric.

Problem 4 Exam 2019

Fix $m \in \mathbb{N}$, and consider the finite-dimensional vector space $\mathbb{C}^{m\times m}$ equipped with the inner product

$$\langle A, B\rangle := \operatorname{tr}(B^H A) = \sum_{j,k=1}^{m} A_{jk}\overline{B_{jk}}, \quad A, B \in \mathbb{C}^{m\times m}. \tag{1}$$

For $\ell, n \in \{1, \dots, m\}$, let $E^{(\ell,n)} \in \mathbb{C}^{m\times m}$ be the matrix whose entry (ℓ, n) has value 1, and all other entries have value 0, i.e.,

$$E^{(\ell,n)}_{jk} = \begin{cases} 1, & \text{if } j = \ell \text{ and } k = n \\ 0, & \text{else} \end{cases}, \quad \text{for all } (j,k) \in \{1, \dots, m\}.$$

a) Show that the set $\mathcal{E} := \{E^{(\ell,n)} : \ell, n \in \{1, \dots, m\}\}$ is an orthonormal basis for $\mathbb{C}^{m\times m}$ with respect to the inner product (1). What is the dimension of the vector space $\mathbb{C}^{m\times m}$?

Next, define the $m\times m$ cyclic time-shift matrix D and the $m\times m$ modulation matrix M according to:

$$D = \begin{pmatrix} 0 & & & \cdots & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} e^{-\frac{2\pi i\cdot 0}{m}} & & & & \\ & e^{-\frac{2\pi i\cdot 1}{m}} & & 0 & \\ & & e^{-\frac{2\pi i\cdot 2}{m}} & & \\ & 0 & & \ddots & \\ & & & & e^{-\frac{2\pi i(m-1)}{m}} \end{pmatrix}.$$

Now, for $\ell, n \in \{0, 1, \dots, m-1\}$, let $G^{(\ell,n)} = \frac{1}{\sqrt{m}} M^\ell D^n$.

b) Show that the set $\mathcal{G} := \{G^{(\ell,n)} : \ell, n \in \{0, 1, \dots, m-1\}\}$ is an orthonormal basis for $\mathbb{C}^{m\times m}$ with respect to the inner product (1).
[Hint: Use a dimensionality argument to establish completeness.]

Given orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{C}^{m\times m}$, we define their mutual coherence $\mu(\mathcal{B}_1, \mathcal{B}_2)$ as

$$\mu(\mathcal{B}_1, \mathcal{B}_2) = \max_{U\in\mathcal{B}_1, V\in\mathcal{B}_2} |\langle U, V\rangle|.$$

c) Compute $\mu(\mathcal{E}, \mathcal{G})$.

d) Show that

$$\mu(\mathcal{B}_1, \mathcal{B}_2) \geq \frac{1}{m},$$

for every pair of orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{C}^{m\times m}$.
[Hint: Assume that the claim is false for orthonormal bases $\mathcal{B}_1 = \{U_1, \dots, U_{m^2}\}$ and

$\mathcal{B}_2 = \{V_1, \ldots, V_{m^2}\}$, and then use the energy conservation identity $\|U_1\|^2 := \langle U_1, U_1 \rangle = \sum_{n=1}^{m^2} |\langle U_1, V_n \rangle|^2$ to derive a contradiction.]

Problem 5 Nonuniform sampling. ☕☕

Let $\{g_k\}_{k=1}^N$ and $\{h_j\}_{j=1}^N$ be orthonormal bases for $\mathbb{C}^N$. Let $\mathcal{P}$ be a probability distribution on $[N]$ such that

$$p_n := \mathbb{P}_{\boldsymbol{t} \sim \mathcal{P}} (\boldsymbol{t} = n) > 0, \quad \text{for all } n \in [N],$$

but otherwise arbitrary. Fix an arbitrary s-sparse vector $x \in \mathbb{C}^N$ with $\|x\|_2 = 1$, and, for $n \in [N]$, define the row vectors

$$Y_n = \left(p_n^{-\frac{1}{2}} \langle g_1, h_n \rangle,\, p_n^{-\frac{1}{2}} \langle g_2, h_n \rangle,\, \ldots,\, p_n^{-\frac{1}{2}} \langle g_N, h_n \rangle \right) \in \mathbb{C}^N$$

as well as the positive numbers

$$\tilde{\mu}_n := \max_{k \in [N]} |\langle g_k, h_n \rangle|.$$

a) Establish $|\langle Y_n, x \rangle|^2 \leq s p_n^{-1} \, \tilde{\mu}_n^2$, for $n \in [N]$.

b) Compute $\mathbb{E}_{\boldsymbol{t} \sim \mathcal{P}} \left[Y_{\boldsymbol{t}}^{\mathrm{H}} \, Y_{\boldsymbol{t}} \right] \in \mathbb{C}^{N \times N}$.

c) Compute $\mathbb{E}_{\boldsymbol{t} \sim \mathcal{P}} \left[|\langle Y_{\boldsymbol{t}}, x \rangle|^2 \right]$.