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Mathematics of Information

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Problem Set 7

Problem 1 Landau's rate in multiband sampling. ☕☕

Recall that Landau's result states that, in order to reconstruct a signal x stably, one must have

$$\mathcal{D}^-(\mathcal{P}) \geq |\mathcal{I}|,$$

where $\mathcal{I}$ is the spectral occupancy of the signal x and $\mathcal{D}^-(\mathcal{P})$ is the lower Beurling density of the sampling set $\mathcal{P}$.

a) Compute the lower Beurling density $\mathcal{D}^-(\mathcal{P}_u)$ for uniform sampling $\mathcal{P}_u \coloneqq \{nd_0\}_{n\in\mathbb{Z}}$.

b) Show that, for a signal x bandlimited to B, i.e., such that $\hat{x}(f) = 0$ for $|f| > B$, Landau's bound with uniform sampling $\mathcal{P}_u$ matches the bound obtained from the sampling theorem.

c) Compute the lower Beurling density $\mathcal{D}^-(\mathcal{P}_p)$ for periodic sampling

$$\mathcal{P}_p \coloneqq \{d_0(n+\tau_1), \dots, d_0(n+\tau_N)\}_{n\in\mathbb{Z}},$$

where N is an integer and $0 \leq \tau_i < 1$, for $i = 1, \dots, N$, and $\tau_i \neq \tau_j$ for $i \neq j$.

Problem 2 Recovery (Exam 2019, Problem 3). ☕☕☕

We consider functions of the form

$$x = \sum_{n\in\mathbb{Z}} c_n\, \phi(\,\cdot - nT), \tag{1}$$

where $T > 0$, $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and $\phi : \mathbb{R} \to \mathbb{R}$ are such that (1) converges unconditionally in $L^2(\mathbb{R})$, but otherwise arbitrary. In many cases such functions can be recovered from samples taken at integer multiples of T, even though they do not need to be bandlimited.

a) Fix $T > 0$ and consider the function

$$x(t) = \begin{cases} 1, & 0 \leq t < T \\ 2, & T \leq t < 2T \\ 1, & 2T \leq t < 3T \\ 4, & 3T \leq t < 4T \\ 0, & \text{else} \end{cases}, \quad t \in \mathbb{R}. \tag{2}$$

(i) Sketch x on the interval $[-T, 5T]$, and show that x can be written in the form (1) by specifying suitable $\{c_n\}_{n\in\mathbb{Z}}$ and ϕ.

(ii) By explicitly computing the Fourier transform of x in (2), show that this x is not bandlimited. You may use — without proof — the fact that the set of zeros of a trigonometric polynomial is discrete.

(iii) Note that x can be reconstructed from the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ taken at integer multiples of T (provided that ϕ is known). Seeing that x is not bandlimited, explain why this does not contradict the sampling theorem.

b) Fix $T > 0$ and let $\phi \in L^2(\mathbb{R})$ be such that $\{\phi(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Furthermore, suppose that ϕ satisfies the following condition:

$$\text{there exists an } \alpha > 0 \text{ s.t. } \Big|\sum_{n\in\mathbb{Z}} \phi(nT)\, e^{-in\theta}\Big| \geq \alpha, \qquad \text{for all } \theta \in [0, 2\pi). \tag{$*$}$$

Now, consider functions x of the form (1) with $\{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(i) Let $\boldsymbol{x} = \{x(nT)\}_{n\in\mathbb{Z}}$. Find elements $\boldsymbol{\phi}^n$ of $\ell^2(\mathbb{Z})$, for $n \in \mathbb{Z}$, such that

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}} c_n\, \boldsymbol{\phi}^n. \tag{3}$$

(ii) Provide an expression for the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ given $\{\phi(nT)\}_{n\in\mathbb{Z}}$ and the samples $\{x(nT)\}_{n\in\mathbb{Z}}$. You may use — without proof — the fact that the series (3) converges unconditionally, and that $\{x(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.