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Mathematics of Information

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Solutions to Problem Set 7

Problem 1 Landau's rate in multiband sampling.

a) We first check that $\mathcal{P}_u \cap [t, t+r] = \{n_1 d_0, \ldots, n_N d_0\}$, where N is an integer such that $n_1 = \left\lceil \frac{t}{d_0} \right\rceil$ and $n_N = \left\lfloor \frac{t+r}{d_0} \right\rfloor$. This is obtained by observing that, for $n_1 d_0$ to be the first elements of $\mathcal{P}_u$ in $[t, t+r]$, n_1 must be such that $(n_1 - 1)d_0 < t \le n_1 d_0$, i.e., $n_1 = \left\lceil \frac{t}{d_0} \right\rceil$. Likewise, in order for $n_N d_0$ to be the last element of $\mathcal{P}_u$ in $[t, t+r]$, n_N must be chosen such that $n_N d_0 \le t + r < (n_N + 1)d_0$, which gives $n_N = \left\lfloor \frac{t+r}{d_0} \right\rfloor$.

Therefore, one has

$$C_r(t) := |\mathcal{P}_u \cap [t, t+r]| = \left\lfloor \frac{t+r}{d_0} \right\rfloor - \left\lceil \frac{t}{d_0} \right\rceil + 1.$$

From the identity

$$C_r(t + kd_0) = \left\lfloor \frac{t + kd_0 + r}{d_0} \right\rfloor - \left\lceil \frac{t + kd_0}{d_0} \right\rceil + 1 = \left\lfloor \frac{t+r}{d_0} \right\rfloor + k - \left\lceil \frac{t}{d_0} \right\rceil - k + 1 = C_r(t),$$

which holds for any $k \in \mathbb{Z}$, one can deduce that C_r is a d_0-periodic function. Therefore, it is sufficient to consider $t \in [0, d_0)$. In particular, $\left\lceil \frac{t}{d_0} \right\rceil \le 1$ and $\left\lfloor \frac{t+r}{d_0} \right\rfloor \ge \left\lfloor \frac{r}{d_0} \right\rfloor$ and both inequalities are achievable. This implies

$$\inf_{t \in \mathbb{R}} C_r(t) = \inf_{t \in [0, d_0)} C_r(t) = \left\lfloor \frac{r}{d_0} \right\rfloor,$$

and therefore

$$\mathcal{D}^-(\mathcal{P}_u) = \lim_{r \to \infty} \inf_{t \in \mathbb{R}} \frac{C_r(t)}{r} = \frac{1}{d_0}.$$

b) Landau's bound reads as

$$\mathcal{D}^-(\mathcal{P}_u) \ge |\mathcal{I}|.$$

In the case of a bandlimited signal, the spectral occupancy $\mathcal{I}$ is the interval $[-B, B]$. For uniform sampling with period d_0, the sampling frequency is $1/d_0$. Applying the result of the previous subproblem, Landau's bound then becomes

$$\frac{1}{d_0} \ge 2B.$$

We indeed recover the bound from the sampling theorem.

c) We first observe that

$$\mathcal{P}_p = \bigcup_{i=1}^{N} \mathcal{P}_{u_i},$$

where $\mathcal{P}_{u_i} = \{d_0(n+\tau_i)\}_{n\in\mathbb{Z}}$ is a uniform sampling set. From the assumption that $\tau_i \neq \tau_j$ whenever $i \neq j$, one can deduce that the sets $\mathcal{P}_{u_i}$, for $i = 1, \ldots, N$, are disjoint. Therefore, using a similar computation as in subproblem a), one obtains

$$\begin{aligned} C_r(t) &\coloneqq |\mathcal{P}_p \cap [t, t+r]| \\ &= \sum_{i=1}^{N} |\mathcal{P}_{u_i} \cap [t, t+r]| \\ &= \sum_{i=1}^{N} |\mathcal{P}_u \cap [t - \tau_i d_0, t - \tau_i d_0 + r]| \\ &= \sum_{i=1}^{N} \left\lfloor \frac{t - \tau_i d_0 + r}{d_0} \right\rfloor - \left\lceil \frac{t - \tau_i d_0}{d_0} \right\rceil + 1. \end{aligned}$$

Again using the same periodicity argument as in subproblem a), one gets

$$N \left\lfloor \frac{r}{d_0} \right\rfloor = \sum_{i=1}^{N} \inf_{t\in\mathbb{R}} \left\lfloor \frac{t - \tau_i d_0 + r}{d_0} \right\rfloor - \left\lceil \frac{t - \tau_i d_0}{d_0} \right\rceil + 1 \leq \inf_{t\in\mathbb{R}} C_r(t),$$

and since one has

$$C_r(t) = \sum_{i=1}^{N} \left\lfloor \frac{t - \tau_i d_0 + r}{d_0} \right\rfloor - \left\lceil \frac{t - \tau_i d_0}{d_0} \right\rceil + 1 \leq \sum_{i=1}^{N} \left\lfloor \frac{r}{d_0} \right\rfloor + 1 = N \left(\left\lfloor \frac{r}{d_0} \right\rfloor + 1 \right),$$

the following estimation holds

$$N \left\lfloor \frac{r}{d_0} \right\rfloor \leq \inf_{t\in\mathbb{R}} C_r(t) \leq N \left(\left\lfloor \frac{r}{d_0} \right\rfloor + 1 \right).$$

This yields

$$\mathcal{D}^-(\mathcal{P}_p) = \lim_{r\to\infty} \inf_{t\in\mathbb{R}} \frac{C_r(t)}{r} = \frac{N}{d_0}.$$

Problem 2 Recovery (Exam 2019, Problem 3).

a) (i) We can write $x = \sum_{n\in\mathbb{Z}} c_n \, \phi(\,\cdot - nT)$, where $\phi = \mathbb{1}_{[0,T)}$, and

$$c_n = \begin{cases} 1, & n \in \{0, 2\} \\ 2, & n = 1 \\ 4, & n = 3 \\ 0, & \text{else} \end{cases}.$$

(ii) The Fourier transform of $\phi = \mathbb{1}_{[0,T)}$ is given by

$$\begin{aligned}\hat{\phi}(\omega) &= \int_0^T e^{-2\pi i\omega t}\mathrm{d}t = \frac{e^{-2\pi i t\omega}}{-2\pi i\omega}\bigg|_{t=0}^{T} = \frac{1-e^{-2\pi i T\omega}}{2\pi i\omega} \\ &= Te^{-\pi i T\omega}\frac{e^{\pi i T\omega}-e^{-\pi i T\omega}}{2\pi i T\omega} = Te^{-\pi i T\omega}\mathrm{sinc}(T\omega), \quad \omega\in\mathbb{R},\end{aligned} \tag{1}$$

where $\mathrm{sinc}(\theta) := \frac{\sin(\pi\theta)}{\pi\theta}$, $\theta \in \mathbb{R}$. Therefore, as x is a linear combination of time-shifted versions of ϕ, we have

$$\hat{x}(\omega) = \underbrace{\left(\sum_{n=0}^{3} c_n\, e^{-2\pi i n T\omega}\right)}_{p(\omega)} \cdot \hat{\phi}(\omega), \quad \omega\in\mathbb{R}.$$

Note that $\hat{x}(\omega) = 0$ if and only if at least one of $p(\omega)$ and $\hat{\phi}(\omega)$ is zero. We have $\{\omega\in\mathbb{R} : \hat{\phi}(\omega) = 0\} = \{\frac{n}{T}\}_{n\in\mathbb{Z}\setminus\{0\}}$ from the explicit expression (1). Moreover, as $p(\omega)$ is a non-zero trigonometric polynomial, the set $\{\omega\in\mathbb{R} : p(\omega) = 0\}$ is discrete. Therefore

$$\{\omega\in\mathbb{R} : \hat{x}(\omega) = 0\} = \{\omega\in\mathbb{R} : \hat{\phi}(\omega) = 0\} \cup \{\omega\in\mathbb{R} : p(\omega) = 0\}$$

is discrete, and hence x is not bandlimited.

(iii) The fact that x can be reconstructed by sampling it at integer multiples of T even though it is not bandlimited does not contradict the sampling theorem as the sampling theorem only states that bandlimitedness is sufficient for reconstruction, but does not claim necessity.

b) (i) Note that

$$x(kT) = \sum_{n\in\mathbb{Z}} c_n\phi(kT-nT), \quad \text{for all } k\in\mathbb{Z},$$

so we simply set $\boldsymbol{\phi}^n = \{\phi((k-n)T)\}_{k\in\mathbb{Z}}$ to obtain

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}} c_n\,\boldsymbol{\phi}^n. \tag{2}$$

(ii) Note that, as $\mathrm{DTFT} : \ell^2(\mathbb{Z}) \to L^2[0,2\pi)$ is continuous and (2) converges unconditionally, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta} &= \mathrm{DTFT}\{\boldsymbol{x}\}(\theta) = \mathrm{DTFT}\Big\{\sum_{n\in\mathbb{Z}} c_n\,\boldsymbol{\phi}^n\Big\}(\theta) \\ &= \sum_{n\in\mathbb{Z}} c_n\,\mathrm{DTFT}\{\boldsymbol{\phi}^n\}(\theta) \\ &= \sum_{n\in\mathbb{Z}} c_n e^{-in\theta}\,\mathrm{DTFT}\{\boldsymbol{\phi}^0\}(\theta) \\ &= \mathrm{DTFT}\{\{c_n\}_{n\in\mathbb{Z}}\}(\theta)\cdot\sum_{k\in\mathbb{Z}}\phi(kT)e^{-ik\theta}, \quad \theta\in[0,2\pi).\end{aligned} \tag{3}$$

Now, let $\alpha > 0$ be such that $\left|\sum_{k\in\mathbb{Z}}\phi(kT)\,e^{-ik\theta}\right| \geq \alpha > 0$, for all $\theta\in[0,2\pi)$, as per the problem assumptions. We can then divide both sides of (3) to obtain

$$\mathrm{DTFT}\{\{c_n\}_{n\in\mathbb{Z}}\}(\theta) = \frac{\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta}}{\sum_{k\in\mathbb{Z}} \phi(kT)e^{-ik\theta}}, \quad \theta \in [0, 2\pi).$$

Finally, inverting the discrete-time Fourier transform, we find

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_{k\in\mathbb{Z}} x(kT)e^{-ik\theta}}{\sum_{k\in\mathbb{Z}} \phi(kT)e^{-ik\theta}}\, e^{in\theta}\mathrm{d}\theta, \quad n \in \mathbb{Z}.$$