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Mathematics of Information

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Problem Set 8

Problem 1 Equivalent Expression for the Restricted Isometry Constants. ☕

Let $A \in \mathbb{C}^{m\times N}$. Show that the restricted isometry constants δ_s of A can be equivalently expressed as

$$\delta_s = \sup_{S\subset[N],\#(S)\le s} \left\|\left\| A_S^{\mathrm{H}} A_S - \mathrm{Id} \right\|\right\|_2 .$$

You may use–without proof–the fact that the operator norm of a hermitian matrix $M \in \mathbb{C}^{n\times n}$ can be expressed as

$$|||M|||_2 = \sup_{x\in\mathbb{C}^n,\|x\|_2\le 1} |\langle Mx, x\rangle|.$$

Problem 2 Properties of the Restricted Isometry Constants. ☕

Let $A \in \mathbb{C}^{m\times N}$ have normalized columns, i.e., $\|A_i\|_2 = 1$, for all $i \in \{1,\dots,N\}$.

a) Prove that $\delta_1 = 0$.

b) Prove that $\delta_2 = \mu(A)$, where $\mu(A) := \max_{i\neq j} |\langle A_i, A_j\rangle|$ is the coherence of A.

c) Suppose that $u, v \in \mathbb{C}^N$ are s-sparse and $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \varnothing$. Show that then

$$|\langle Au, Av\rangle| \le \delta_{2s}\|u\|_2\|v\|_2.$$

Problem 3 Restricted Orthogonality (Exam 2020/2021, Problem 1). ☕☕

In this problem, we define the *(s,t)-restricted orthogonality constant* $\theta_{s,t} = \theta_{s,t}(A)$ of a matrix $A \in \mathbb{C}^{m\times N}$ as the smallest $\theta \ge 0$ such that

$$|\langle Au, Av\rangle| \le \theta\|u\|_2\|v\|_2$$

for all disjointly supported s-sparse and t-sparse vectors $u \in \mathbb{C}^N$ and $v \in \mathbb{C}^N$, respectively. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{C}^N$ and $m, N \in \mathbb{N}$.

Moreover, for a vector $u \in \mathbb{C}^N$, a matrix $B \in \mathbb{C}^{m\times N}$, and a set $S \subset \{1,\dots,N\}$, we define $u_S \in \mathbb{C}^{|S|}$ to be the vector obtained from u by keeping only the entries indexed by S, and similarly, we define $B_S \in \mathbb{C}^{m\times|S|}$ to be the matrix obtained from B by keeping only the columns indexed by S.

a) Show that

$$\theta_{s,t} = \max\left\{ \|A_T^H A_S\|_2 \mid S, T \subset \{1, \dots, N\},\ S \cap T = \emptyset,\ |S| \le s,\ |T| \le t \right\},$$

where $\|\cdot\|_2$ denotes the matrix operator norm with respect to the ℓ^2-norm on $\mathbb{C}^{|S|}$, i.e.,

$$\|A_T^H A_S\|_2 := \max_{\substack{u \in \mathbb{C}^{|S|} \\ \|u\|_2 \le 1}} \|(A_T^H A_S)u\|_2.$$

b) Prove the following relation between the restricted isometry constant and the restricted orthogonality constant

$$\theta_{s,t} \le \delta_{s+t}.$$

c) In this subproblem, we want to show that

$$\theta_{t,r} \le \sqrt{\frac{t}{s}}\, \theta_{s,r},$$

where $r, s, t \ge 1$ are such that $t \ge s$. To this end, let $u \in \mathbb{C}^N$ be t-sparse, $v \in \mathbb{C}^N$ r-sparse and u and v are disjointly supported. Furthermore, let $T = \{j_1, j_2, \dots, j_t\}$ denote the support set of u and consider the t subsets $S_1, S_2, \dots, S_t \subset T$ of cardinality s defined by

$$S_i = \{j_i, j_{i+1}, \dots, j_{i+s-1}\}, \quad \text{for all} \quad i \in \{1, 2, \dots, t\},$$

where the indices are understood to be modulo t.

i) Show that

$$u = \frac{1}{s} \sum_{i=1}^{t} u_{S_i} \quad \text{and} \quad \|u\|_2^2 = \frac{1}{s} \sum_{i=1}^{t} \|u_{S_i}\|_2^2.$$

ii) Use c)i) to establish that

$$|\langle Au, Av \rangle| \le \sqrt{\frac{t}{s}}\, \theta_{s,r}\, \|u\|_2 \|v\|_2.$$

Problem 4 Johnson-Lindenstrauss with sub-Gaussian Matrices.

We recall that a sub-Gaussian random variable Z with parameter $\sigma > 0$ is defined as a random variable satisfying

$$\mathbb{E}\left[e^{\lambda Z}\right] \le e^{\frac{\lambda^2 \sigma^2}{2}}, \quad \forall \lambda \in \mathbb{R},$$

and that a sub-Gaussian random variable Z with parameter 1 satisfies

$$\mathbb{E}\left[e^{\lambda Z^2}\right] \le \frac{1}{\sqrt{1 - 2\lambda}}, \quad \forall \lambda \in [0, 1/2).$$

Show that the Johnson-Lindenstrauss Lemma can be proven using sub-Gaussian matrices. More precisely, fix $0 < \epsilon < 1$ together with k and n two integers, and let $\boldsymbol{A} \in \mathbb{R}^{k \times n}$ be a random matrix, with entries taken to be i.i.d. sub-Gaussian random variables with parameter $1/\sqrt{k}$. Then, prove that, for every $\boldsymbol{u} \in \mathbb{R}^n$, we have

$$\mathbb{P}\left[\left|\|\boldsymbol{Au}\|^2 - \|\boldsymbol{u}\|^2\right| \geq \epsilon\|\boldsymbol{u}\|^2\right] < 2e^{-k\frac{\epsilon^2-\epsilon^3}{4}}.$$

Problem 5 Random Matrices (Exam 2020, Problem 1).

Let $\{g_k\}_{k=1}^N$ and $\{h_j\}_{j=1}^N$ be orthonormal bases for $\mathbb{C}^N$, and let $\mathcal{P}$ be a probability distribution on $[N] := \{1, 2, \ldots, N\}$ such that

$$p_n := \mathbb{P}_{\boldsymbol{t}\sim\mathcal{P}}(\boldsymbol{t} = n) > 0, \quad \text{for all } n \in [N],$$

but otherwise arbitrary, where $\boldsymbol{t}$ denotes a random variable taking values in $[N]$. Furthermore, set

$$\tilde{\mu}_j := \max_{k\in[N]} |\langle g_k, h_j\rangle|, \quad \text{for } j \in [N],$$

and

$$\kappa := \max_{j\in[N]} p_j^{-1}\tilde{\mu}_j^2.$$

For a set of scalar or vector quantities u_j indexed by $j \in [N]$, we write $u_{\boldsymbol{t}}$ for the random variable taking on the value u_n on the event $\{\boldsymbol{t} = n\}$, for each $n \in [N]$.

a) We define a probability distribution $\widetilde{\mathcal{P}}$ on $[N]$ according to

$$\mathbb{P}_{\boldsymbol{t}\sim\widetilde{\mathcal{P}}}(\boldsymbol{t} = n) = \frac{\tilde{\mu}_n^2}{\tilde{\mu}_1^2 + \cdots + \tilde{\mu}_N^2}, \quad \text{for } n \in [N].$$

   i) Show that $\sum_{j=1}^N \tilde{\mu}_j^2 \geq 1$.

   ii) Show that $\kappa^{-1} \leq \mathbb{E}_{\boldsymbol{t}\sim\widetilde{\mathcal{P}}}\left[p_{\boldsymbol{t}}\,\tilde{\mu}_{\boldsymbol{t}}^{-2}\right]$.

   iii) Use the inequality in a)ii) to show that $\kappa \geq \sum_{j=1}^N \tilde{\mu}_j^2$, with equality if and only if $\mathcal{P} = \widetilde{\mathcal{P}}$.

b) Now, let $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m$ be independent samples drawn from the distribution $\mathcal{P}$ and define a random matrix $A \in \mathbb{C}^{m\times N}$ according to $A_{\ell k} = (mp_{\boldsymbol{t}_\ell})^{-\frac{1}{2}}\langle g_k, h_{\boldsymbol{t}_\ell}\rangle$, for $\ell \in [m]$, $k \in [N]$. Furthermore, let $x \in \mathbb{C}^N$ be an arbitrary s-sparse vector with $\|x\|_2 = 1$ and, for $\ell \in [m]$, define

$$X_\ell = |\langle x, Y_\ell\rangle|^2 - \tfrac{1}{m},$$

where $Y_\ell = (\overline{A}_{\ell 1}\, \overline{A}_{\ell 2}\, \cdots\, \overline{A}_{\ell N})$ is the complex conjugate of the ℓ^{th} row of A.

   i) Show that $\mathbb{E}[X_\ell] = 0$, $|X_\ell| \leq sm^{-1}\kappa$, and $\mathbb{E}\left[|X_\ell|^2\right] \leq sm^{-2}\kappa$.

   ii) Use Bernstein's inequality together with the results of part b)i) to establish

$$\mathbb{P}\left(\left|\|Ax\|_2^2 - 1\right| \geq t\right) \leq 2\exp\left(-\frac{3t^2m}{8s\kappa}\right), \quad \text{for } t \in (0, 1).$$