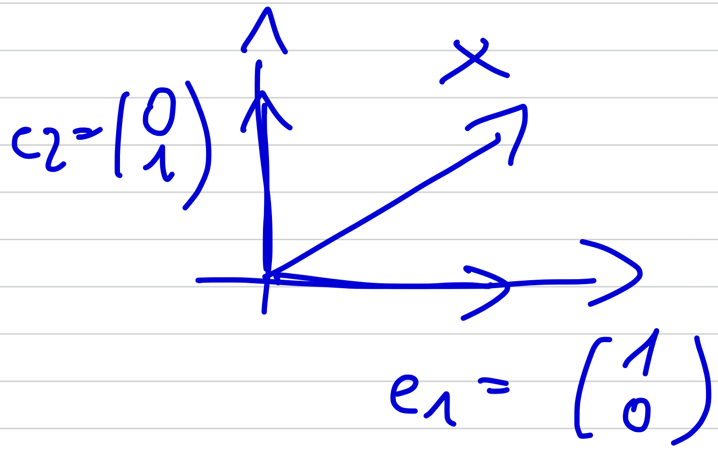
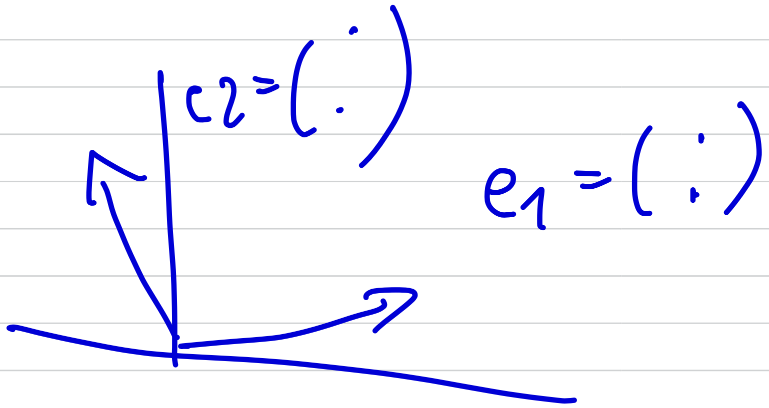
Frame theory

$\mathbb{R}^2$



$\langle e_1, e_2 \rangle = 0$

ONB



$$x = \underbrace{\langle x, e_1 \rangle}\, e_1 + \langle x, e_2 \rangle\, e_2$$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x = \underbrace{\langle x, e_1 \rangle}_{c_1} e_1 + \underbrace{\langle x, e_2 \rangle}_{c_2} e_2$$

$$= x_1 e_1 + x_2 e_2$$

$$x = c_1 e_1 + c_2 e_2$$

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix}}_{2 \times 2} x = \begin{pmatrix} \langle x, e_1 \rangle \\ \langle x, e_2 \rangle \end{pmatrix}$$

$$x \xrightarrow{T} c$$

$$c = Tx$$

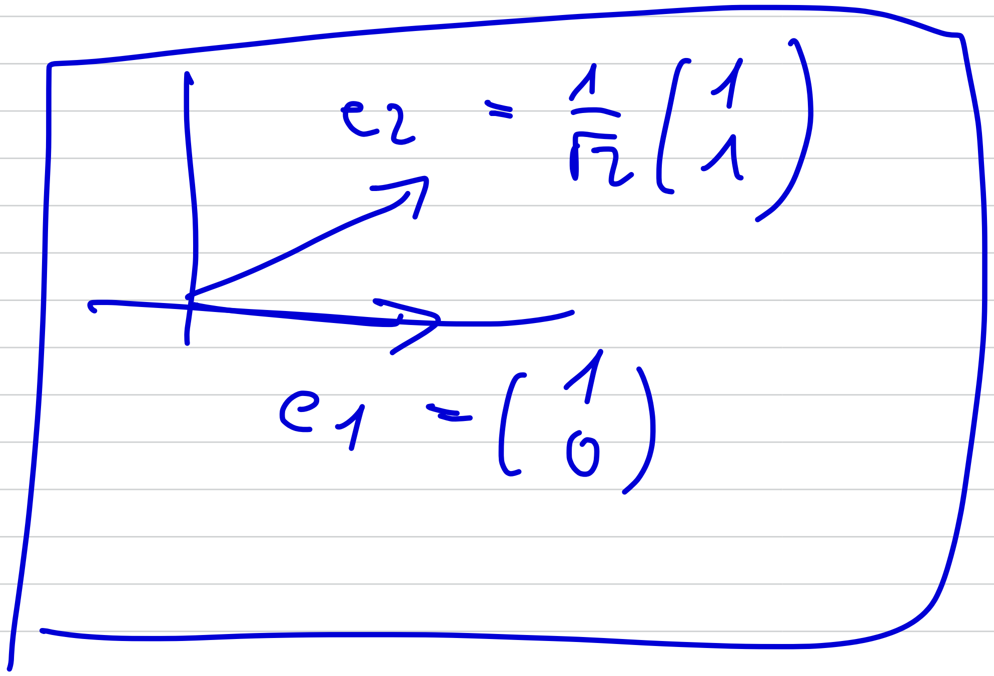
$$T^T c = (e_1 \quad e_2)\, c = (e_1 \quad e_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e_1 + c_2 e_2$$

$$c = \underbrace{T}_{\text{analysis operator}} x \quad \text{... analysis}$$

$$x = \underbrace{T^T}_{\text{synthesis operator}} c \quad \text{... synthesis}$$

$$T = \begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$T^T = (e_1 \quad e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$



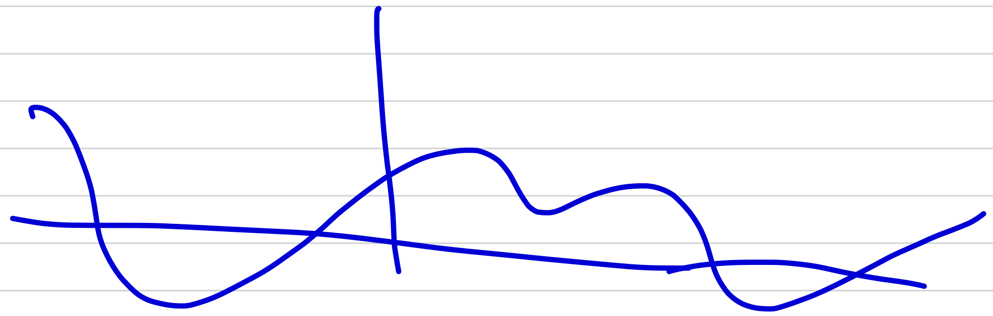
Fourier series

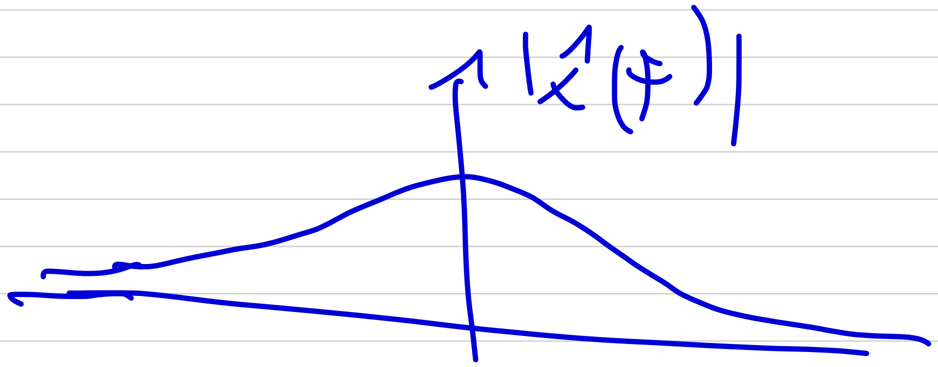
$$x(t + T) = x(t)$$

$$x(t) = \sum_{l=-\infty}^{\infty} c_l e^{i 2\pi l \frac{t}{T}}$$

$$c_l = \frac{1}{T} \int_0^T x(t) e^{-i 2\pi l \frac{t}{T}} dt$$

$$e_l(t) = e^{i 2\pi l \frac{t}{T}}$$





$$c_\ell = \langle x, e_\ell \rangle$$

$$x(t) = \sum_\ell \langle x, e_\ell \rangle e_\ell(t)$$

Fourier transform

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \, dt$$

$$= \langle x(\cdot), e^{i2\pi f \cdot} \rangle$$

Can we expand ~~any~~ every $x \in \mathbb{R}^2$ into $\{e_1, e_2\}$ with e_1 & e_2 not collinear?

$$x = \underline{\langle x, e_1 \rangle \tilde{e}_1 + \langle x, e_2 \rangle \tilde{e}_2}$$

$$= \underbrace{(\tilde{e}_1 \;\; \tilde{e}_2)}_{\tilde{T}^T} \underbrace{\begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix}}_{T \in \mathbb{R}^{2\times 2}} x$$

$$x = \underbrace{\underbrace{(\tilde{e}_1 \quad \tilde{e}_2)}_{\tilde{T}^T} \underbrace{\begin{pmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{T}} x$$

$$\underbrace{\tilde{T}^T}_{=T^{-1}} T = I_2$$

$$\begin{pmatrix} 1 & 0 \\ \underbrace{-1}_{\tilde{e}_1} & \underbrace{\sqrt{2}}_{\tilde{e}_2} \end{pmatrix}$$

$$\underbrace{T^{-1}}_{\tilde{T}^T} \underbrace{T}_{T} = \cancel{T \, T^{-1}} = I_2$$

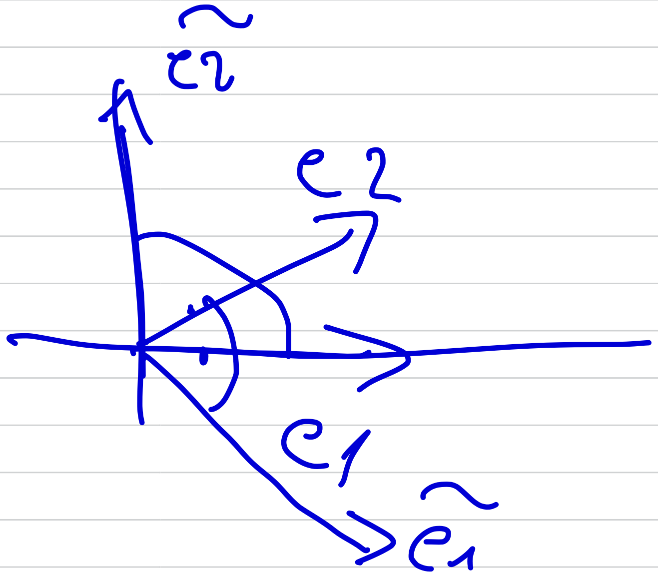
$$T \, T^{-1} = \begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix} (\tilde{e}_1 \quad \tilde{e}_2) = \begin{pmatrix} \langle e_1, \tilde{e}_1 \rangle & \langle e_1, \tilde{e}_2 \rangle \\ \langle e_2, \tilde{e}_1 \rangle & \langle e_2, \tilde{e}_2 \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

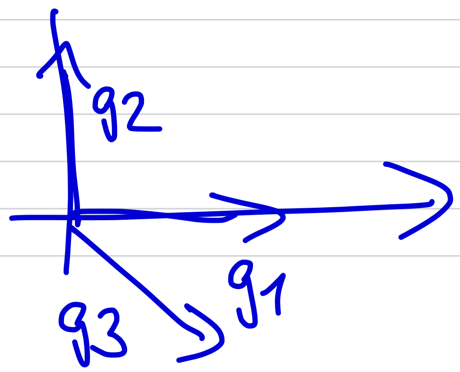
$\langle e_1, \tilde{e}_1 \rangle = 1$ $\quad \langle e_2, \tilde{e}_1 \rangle = 0$

$\langle e_1, \tilde{e}_2 \rangle = 0$ $\quad \langle e_2, \tilde{e}_2 \rangle = 1$

biorthonormality



redundant signal sets



$g_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

$g_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\underbrace{c}_{3\times 1} = T x = \underbrace{\begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \end{pmatrix}}_{3\times 2} \underbrace{x}_{2\times 1} \qquad T^T = (\underbrace{g_1 \quad g_2}_{\text{ONB}} \quad g_3)$$

$$x = \langle x, g_1\rangle \tilde{g}_1 + \langle x, g_2\rangle \tilde{g}_2 + \langle x, g_3\rangle \tilde{g}_3$$

One possibility is : $\tilde{g}_3 = 0,\ \tilde{g}_1 = g_1,\ \tilde{g}_2 = g_2$

$$\underline{x = \langle x, g_1\rangle g_1 + \langle x, g_2\rangle g_2}$$

$$x = \langle x, g_1\rangle g_1 + \langle x, g_2\rangle g_2 + \underbrace{\langle x, g_1 - g_2\rangle (g_1 - g_1)}_{=0}$$

$$= \langle x, g_1\rangle \underbrace{2g_1}_{\tilde{g}_1} + \langle x, g_2\rangle \underbrace{(g_2 - g_1)}_{\tilde{g}_2} - \langle x, \overbrace{g_1 - g_2}^{g_3}\rangle g_1 \qquad \tilde{g}_3 = -g_1$$

$$= \langle x, g_1\rangle \tilde{g}_1 + \langle x, g_2\rangle \tilde{g}_2 + \langle x, g_3\rangle \tilde{g}_3$$

second possibility: $\tilde{g_1} = 2g_1$, $\tilde{g_2} = g_2 - g_1$, $\tilde{g_3} = -g_1$

$$c = \underset{3 \times 2}{T}\, x$$

$$x = \underset{2 \times 3}{L}\, \underset{3 \times 2}{T}\, x \quad , \quad LT = I_2$$

1.2. Signal expansions in finite-dimensional Hilbert spaces

ONB: The set of vectors $\{e_k\}_{k=1}^{M}$, $e_k \in \mathbb{C}^M$, is called an ONB if

1. $\operatorname{span}\{e_k\}_{k=1}^{M} = \{c_1 e_1 + c_2 e_2 + \dots + c_M e_M \mid c_1, c_2, \dots, c_M \in \mathbb{C}\} = \mathbb{C}^M$

2. $$\langle e_k, e_j \rangle = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}, \quad k, j = 1, \dots, M$$

$$x = \sum_{k=1}^{M} \langle x, e_k \rangle e_k = T^H T x$$

$$T^H = [e_1\ e_2 \dots e_M] \qquad T = \begin{bmatrix} e_1^H \\ e_2^H \\ \vdots \\ e_M^H \end{bmatrix}$$

general bases in $\mathbb{C}^M$, retain spanning property, but relax orthonormality

$$T = \begin{bmatrix} e_1^H \\ e_2^H \\ \vdots \\ e_M^H \end{bmatrix}$$

$$x = \underbrace{\tilde{T}^H T}_{I_M} x \Rightarrow \tilde{T}^H = T^{-1} = [\tilde{e}_1\ \tilde{e}_2 \dots \tilde{e}_M]$$

$$T^{-1}T = T T^{-1} = \begin{bmatrix} e_1^H \\ e_2^H \\ \vdots \end{bmatrix} [\tilde{e}_1 \; \tilde{e}_2 \; \dots \; \tilde{e}_n] = I_n$$

$$I_n = \begin{bmatrix} \langle \tilde{e}_1, e_1 \rangle & \langle \tilde{e}_2, e_1 \rangle & \dots \\ \langle \tilde{e}_1, e_2 \rangle & \langle \tilde{e}_2, e_2 \rangle & \dots \\ \vdots & & \end{bmatrix}$$

$$\langle \tilde{e}_k, e_j \rangle = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$ biorthonormality

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$$x = \langle x, g_1 \rangle g_1 + \langle x, g_2 \rangle g_2 + \underbrace{\langle x, g_1 - g_2 \rangle (g_1 - g_1)}$$

$$\underline{\langle x, g_1 \rangle g_1} - \underline{\langle x, g_1 - g_2 \rangle g_1}$$

Redundant signal expansions in finite-dim. spaces

$$\{g_1, g_2, \ldots, g_N\}, \quad \underline{N > M}$$

$$g_i \in \mathbb{C}^M$$

$$x \in \mathbb{C}^M$$

$$x = \sum_{\ell=1}^{M} \langle x, g_\ell \rangle \tilde{g}_\ell$$

$$T = \underbrace{\begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \\ g_N^H \end{bmatrix}}_{N \times M}$$

$$x = \underbrace{L\,T}_{I_M}\,x$$

$$L = [\tilde{g}_1 \ \tilde{g}_2 \ \ldots \ \tilde{g}_N]$$

$L = (T^H T)^{-1} T^H$ ... Moore-Penrose (pseudo) inverse

$$LT = (T^H T)^{-1} \underbrace{T^H T}_{} = I$$

Theorem 1.6. $A \in \mathbb{C}^{N \times M}$, $N \geq M$, $\operatorname{rank}(A) = M$. $A^+ = (A^H A)^{-1} A^H$

The general solution of the equation $LA = I_M$ is given by

$$L = A^+ + M(I_N - AA^+)$$

where $M \in \mathbb{C}^{M \times N}$ is an arbitrary matrix.

Proof. 1. $L = A^+ + M(I_N - AA^+)$ is a left-inverse

$$LA = (A^+ + M(I_N - AA^+))A =$$

$$= \underbrace{A^+ A}_{I} + M(A - \underbrace{\underbrace{A \underbrace{A^+ A}_{I}}_{(A-A)=0}}_{=0})$$

$$= I$$

2. Every L can be written in the form

$$L = A^+ + M(I - AA^+)$$

$$LA = I_m \quad | \cdot A^+$$

$$LAA^+ = A^+ \quad | + L$$

$$L + LAA^+ = A^+ + L$$

$$L = A^+ + L - LAA^+ = A^+ + \underbrace{L}_{M}(I - AA^+).$$

Parseval: Fourier series : $x(t+T) = x(t)$

$$\underbrace{\frac{1}{T}\int_0^T |x(t)|^2 \, dt}_{\|x\|_2} = \underbrace{\sum_k |c_k|^2}_{\|c\|_{\ell_2}}$$

$$x = \underbrace{T^H T}_{I} x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2$$

$$c = Tx$$

$$\|c\|_2^2 = \|Tx\|_2^2 = x^H \underbrace{T^H T}_{I} x = x^H x = \|x\|_2^2$$

general basis: $\underline{\underline{\|c\|_2^2}} = x^H \underbrace{T^H T}_{} x$

also holds for frames

$$\boxed{\lambda_{min} \|x\|_2^2 \leq x^H \underbrace{T^H T}_{} x \leq \lambda_{max} \|x\|_2^2}$$

$S = T^H T$

$S^H = T^H T = \underline{\underline{S}}$

ONB: $\underline{\|x\|_2^2} = x^H T^H T x = \underline{\|c\|_2^2}$ ... Parseval

1.3. Frames for General Hilbert Spaces

Duffin & Schaeffer , $x \in \mathcal{H}$, $g_k \in \mathcal{H}$

general frame $\{g_k\}_{k \in K}$

Def. 1.7. $T : \mathcal{H} \to \ell^2$

$$\underline{T : x \to \{\langle x, g_k \rangle\}_{k \in K}}$$

$$\underbrace{\begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix}}_{T} x$$

$$\|Tx\|^2 = \sum_k |\langle x, g_k \rangle|^2$$