$$T : x \to \{\langle x, g_k \rangle\}_{k \in K}$$

$$\underbrace{\begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix}}_{T} x$$

$$\|Tx\|^2 = \sum_k |\langle x, g_k \rangle|^2$$

1.3. Frames for general Hilbert spaces

recall: in finite-dim. spaces

$$T = \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix}, \quad c = Tx = \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix} x$$

$$= \begin{bmatrix} \langle x, g_1 \rangle \\ \langle x, g_2 \rangle \\ \vdots \end{bmatrix}$$

Def. 1.7. $T: x \to \{\langle x, g_k\rangle\}_{k \in K}$

$x \in \mathcal{H}$ $\qquad \ell^2$

$c_k = \langle x, g_k\rangle$

$$\sum_{k \in K} |\langle x, g_k\rangle|^2 < \infty$$

$$\|Tx\|^2 = \|c\|^2 < \infty$$

$$\|Tx\|^2 = \sum_{k \in K} |\langle x, g_k\rangle|^2$$

$\Big($ $c = Tx$

$c_1 = Tx_1$

$c_2 = Tx_2$

$\underbrace{c_1 - c_2}_{\neq 0} = Tx_1 - Tx_2$

$= T\underbrace{(x_1 - x_2)}_{\neq 0}$ $\Big)$

1.

$$A\|x-y\|^2 \leq \|Tx - Ty\|^2 \quad , \quad A > 0$$

$$A\|x-y\|^2 = \|T(x-y)\|^2$$

$$A\|z\|^2 \leq \|Tz\|^2 \quad , \forall z \in \mathcal{H}$$

2. $\|Tx\|^2 = \|x\|^2$ ... Parseval (for ONBs)

$$\|Tz\|^2 = \sum_{\lambda \in K} |\langle z, g_\lambda \rangle|^2 \leq B\|z\|^2, \forall z \in \mathcal{H}$$

$$B < \infty$$

(3. "going back from the coefficients to the signal in a numerically stable fashion")

Def. 1.8. A set of elements $\{g_\lambda\}_{\lambda \in K}$, $g_\lambda \in \mathcal{H}$, is called a frame for the Hilbert space $\mathcal{H}$ if

$$A\|x\|^2 \leq \sum_{\lambda} |\langle x, g_\lambda \rangle|^2 \leq B\|x\|^2 \quad , \forall x \in \mathcal{H}$$

with $A, B \in \mathbb{R}$, $0 < A \leq B < \infty$. Valid constants A and B are called frame bounds. The largest valid A and the smallest valid B are called "the" frame bounds.

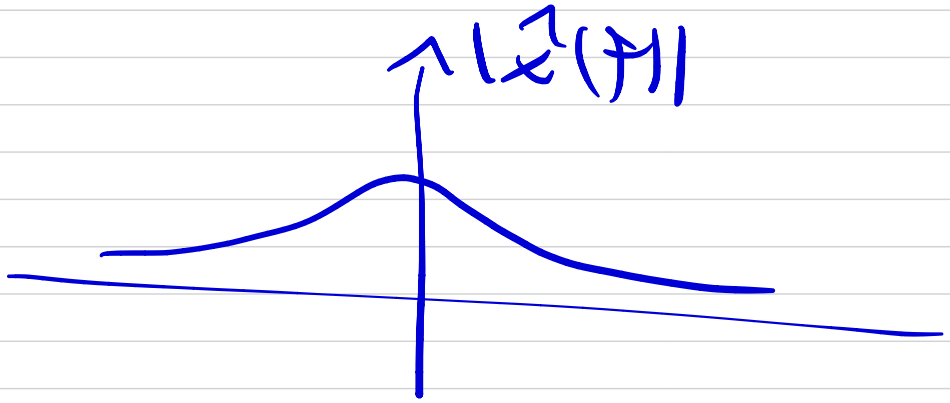
Interlude

$$\left(\hat{x}(f) = \int x(t) \underbrace{e^{-i2\pi f t}}_{\overline{g_f(t)}} dt \right) \qquad g_f(t) = e^{i2\pi f t}, \quad f \in \mathbb{R}$$

continuous frame expansions

$$\underbrace{\hspace{6em}}_{\langle x, g_f \rangle}$$

$$g_{f_0}(t) = e^{i2\pi f_0 t}$$



Fourier series

$$c_k = \frac{1}{T}\int_0^T x(t) \underbrace{e^{-i2\pi k \frac{t}{T}}}_{\overline{g_k(t)}} dt, \qquad x(t) = x(t+T), \ \forall t \in \mathbb{R}$$

$$g_k(t) = e^{i2\pi k \frac{t}{T}}$$

Ex. 1.9. Let $\{e_k\}_{k=1}^{\infty}$ be an ONB for $\mathcal{H}$ (∞-dim.). By repeating each element in $\{e_k\}_{k=1}^{\infty}$, we obtain the redundant set

$$\{g_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}.$$

$$c = \{\underbrace{\langle x, e_1\rangle}_{c_1}, \underbrace{\langle x, e_1\rangle}_{c_1}, \underbrace{\langle x, e_2\rangle}_{c_2}, \underbrace{\langle x, e_2\rangle}_{c_2}, \ldots\}$$

$$\sum_k |\langle x, e_k\rangle|^2 = \|x\|^2 \quad \ldots \text{ Parseval}$$

$$\sum_k |\langle x, g_k\rangle|^2 = \sum_k |\langle x, e_k\rangle|^2 + \sum_k |\langle x, e_k\rangle|^2$$

$$= 2\sum_k |\langle x, e_k\rangle|^2 = 2\|x\|^2$$

$$\sum_k |\langle x, g_k\rangle|^2 = 2\|x\|^2, \quad A = B = 2$$

(tight frame)

Ex. 1.10. ONB $\{e_k\}_{k=1}^{\infty}$, construct a redundant set according to

$$\{g_k\}_{k=1}^{\infty} = \{ \widetilde{e_1}, \overbrace{\tfrac{1}{\sqrt{2}} e_2, \tfrac{1}{\sqrt{2}} e_2}, \overbrace{\tfrac{1}{\sqrt{3}} e_3, \tfrac{1}{\sqrt{3}} e_3, \tfrac{1}{\sqrt{3}} e_3}, \dots \}$$

$$\sum_{k=1}^{\infty} |\langle x, g_k\rangle|^2 = \sum_{k=1}^{\infty} k \left|\langle x, \tfrac{1}{\sqrt{k}} e_k\rangle\right|^2 =$$

$$= \sum_{k=1}^{\infty} k \, \tfrac{1}{k} |\langle x, e_k\rangle|^2 =$$

$$= \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \overset{\text{Parseval}}{=} \|x\|^2$$

$$\sum_{k=1}^{\infty} |\langle x, g_k\rangle|^2 = \|x\|^2 \ , \quad A = B = 1$$

Def. 1.11. $\{g_k\}_{k\in K}$ is complete for $\mathcal{H}$ if $\langle x, g_k\rangle = 0, \forall k \in K$, implies $x = 0$

frame property $\Rightarrow$ completeness

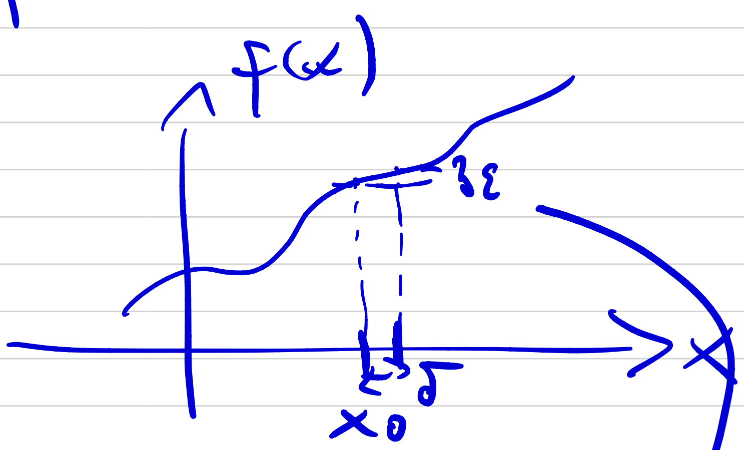
$$A\|x\|^2 \leq \sum_{g \in \mathcal{K}} \underbrace{|\langle x, g\rangle|^2}_{=0} = 0 \quad , \quad A>0$$

$$\Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$$

boundedness of T

$$\sum_{g} |\langle x, g\rangle|^2 \leq B\|x\|^2 \quad , \quad B<\infty$$

continuity of T



$$\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon$$

the equivalent of T^H

$$c = T\,x = \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix} x = \begin{bmatrix} \langle x, g_1 \rangle \\ \langle x, g_2 \rangle \\ \vdots \end{bmatrix}$$

$$T^H c = [g_1 \; g_2 \ldots]\, c = \sum_i c_i g_i$$

Def. 1.13. $A : \mathcal{H} \to \mathcal{H}'$, $A^* : \mathcal{H}' \to \mathcal{H}$ s.t.

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

A^* is the adjoint of A

$$\langle Ax, y \rangle = y^H A x = (A^H y)^H x = \langle x, A^H y \rangle$$

$$\langle Ax, y \rangle = \langle x, A^H y \rangle$$

Def. 1.12. $T^* : \ell^2 \to \mathcal{H}$

$$T^\times : \{c_\ell\}_{\ell \in K} \to \sum_\ell c_\ell g_\ell$$

$$\langle Tx, c\rangle = \langle x, T^\times c\rangle \quad \Rightarrow T^\times = T^*$$

$$\langle Tx, c\rangle = \sum_\ell \langle x, g_\ell\rangle c_\ell^*$$

$$\langle x, T^\times c\rangle = \langle x, \sum_\ell c_\ell g_\ell\rangle = \sum_\ell c_\ell^* \langle x, g_\ell\rangle$$

1.3.1. The frame operator

$$\tilde{g}_\ell = (\underbrace{T^H T}_{S = T^H T})^{-1} g_\ell = S^{-1} g_\ell$$

Def. 1.14. $S : \mathcal{H} \to \mathcal{H}$

$$S = T^* T$$

$$Sx = \sum_\ell \langle x, g_\ell\rangle g_\ell$$

is called the frame operator.

$$T^*Tx = T^*\{\langle x, g_\ell\rangle\} = \sum_\ell \langle x, g_\ell\rangle g_\ell$$

$$A\|x\|^2 \le \underbrace{\overbrace{\sum_\ell |\langle x, g_\ell\rangle|^2}^{\langle Sx, x\rangle}}_{\|Tx\|^2} \le B\|x\|^2$$

(in finite-dim. $\|Tx\|^2 = \langle Tx, Tx\rangle = x^H \underbrace{T^H T}_{S} x$

$$= x^H S x$$

$$= \langle Sx, x\rangle\ )$$

$$\langle Sx, x\rangle = \left\langle \sum_\ell \langle x, g_\ell\rangle g_\ell, x\right\rangle$$

$$= \sum_\ell \langle x, g_\ell\rangle \langle g_\ell, x\rangle$$

$$= \sum_\ell \langle x, g_\ell\rangle \langle x, g_\ell\rangle^*$$

$$= \sum_\ell |\langle x, g_\ell\rangle|^2$$

$$\boxed{A\|x\|^2 \leq \langle Sx, x\rangle \leq B\|x\|^2} \qquad S = T^*T$$

$$S = T^H T$$

$$S^H = (T^H T)^H = T^H T = S$$

$$S S^H = S^H S \Rightarrow \text{eigendecomposition}$$

$$A = U \Sigma V^H$$

Th 1.11. The frame operator satisfies

1. S is linear & bounded
2. S is self-adjoint (i.e., $S^* = S$)
3. S has a unique self-adjoint pos.def. square root (denoted as $S^{1/2}$), $S = S^{1/2} S^{1/2}$

$$S = T^*T, \quad S^* = (T^*T)^* = T^* \underbrace{(T^*)^*}_{T} = T^*T = S$$

Spectral thm. $A = \lambda_{min}(S)$, $B = \lambda_{max}(S)$

$$A\|x\|^2 \le \langle Sx, x\rangle \le B\|x\|^2$$

1.3.2. The canonical dual frame

$$\tilde{g_k} = (T^H T)^{-1} g_k$$

$$L = (T^H T)^{-1} T^H = S^{-1} T^H = S^{-1} [g_1 \; g_2 \ldots]$$

$$LT = \underline{I} \qquad = [S^{-1} g_1 \; S^{-1} g_2 \ldots]$$

$$\{\tilde{g_k} = S^{-1} g_k\}_{k \in K}$$

Th. 1.18. 1. S^{-1} is self-adjoint

2. $\frac{1}{B} = \lambda_{min}(S^{-1})$

$\frac{1}{A} = \lambda_{max}(S^{-1})$

Proof. 1. $\underbrace{(SS^{-1})}_{I}{}^* = (S^{-1})^* \underbrace{S^*}_{S} = I \quad | \cdot S^{-1}$

$$(\langle x, Iy \rangle = \langle I^* x, y \rangle = \langle x, y \rangle)$$

$$(S^{-1})^* = S^{-1} \Rightarrow \text{q.e.d.}$$

$\{\hat{g_k} = S^{-1} g_k\}$ is a frame?

Th. 1.19. 1. The set $\{\hat{g_k} = S^{-1} g_k\}_{k \in K}$ is a frame for $\mathcal{H}$ with the frame bounds $\frac{1}{B}, \frac{1}{A}$

$$\frac{1}{B} \|x\|^2 \leq \sum_k |\langle x, \hat{g_k} \rangle|^2 \leq \frac{1}{A} \|x\|^2$$

2. The analysis op. corr. to $\{\hat{g_k}\}$ is given by

$$\hat{T} = T S^{-1}$$

Proof. $\displaystyle\sum_k |\langle x, \tilde{g_k}\rangle|^2 \overset{\tilde{g_k} = S^{-1} g_k}{=} \sum_k |\langle x, S^{-1} g_k\rangle|^2$

$$= \sum_k |\langle \overbrace{S^{-1}x}^{y}, g_k\rangle|^2$$

$$= \langle S y, y\rangle$$

$$= \langle \underbrace{S S^{-1}}_{I} x, S^{-1} x\rangle$$

$$= \langle x, S^{-1} x\rangle$$

$$= \langle S^{-1} x, x\rangle$$

$$\frac{1}{B}\|x\|^2 \le \underbrace{\sum_k |\langle x, \tilde{g_k}\rangle|^2}_{\langle S^{-1}x, x\rangle} \le \frac{1}{A}\|x\|^2$$

$$(\tilde{T} x)_k = \langle x, \tilde{g_k}\rangle = \langle x, S^{-1} g_k\rangle \overset{S^{-1} = (S^{-1})^*}{=} \langle \underbrace{S^{-1} x}_{y}, g_k\rangle = T y = T S^{-1} x$$

$$\Rightarrow \quad \tilde{T} = T S^{-1}$$

1.3.3. Signal expansions

recall: $x = \sum_{\xi} \langle x, g_\xi \rangle \underline{\tilde{g_\xi}}$

$$\tilde{g_\xi} = (T^* T)^{-1} g_\xi$$

Th. 1.22. $\{g_\xi\}_{\xi \in X}$ & $\{\tilde{g_\xi}\}_{\xi \in X}$ are canonical dual frames.

$$x = T^* \tilde{T} x = \sum_\xi \langle x, \tilde{g_\xi} \rangle g_\xi$$

$$x = \tilde{T}^* T x = \sum_\xi \langle x, g_\xi \rangle \tilde{g_\xi}$$

$$T^* \tilde{T} = \tilde{T}^* T = \underline{I}$$

Proof. $\tilde{T} = T S^{-1}$

$$T^* \tilde{T} = \underbrace{T^* T}_{S} S^{-1} = S S^{-1} = \underline{I}$$

$$\tilde{T}^* \tilde{T} = (TS^{-1})^* T = \underbrace{(S^{-1})^*}_{=S^{-1}} \underbrace{T^* T}_{S} = S^{-1} S = \underline{I}.$$