$$\hat{T}^* T = (TS^{-1})^* T = \underbrace{(S^{-1})^*}_{=S^{-1}} \underbrace{T^* T}_{S} = S^{-1} S = I. \quad \square$$

$$\hat{T} x = \langle x, \hat{g}_k \rangle = \langle x, S^{-1} g_k \rangle = \langle S^{-1} x, g_k \rangle = T S^{-1} x, \quad \forall x \in \mathcal{H}$$

$$\Rightarrow \hat{T} = T S^{-1}$$

in finite dimensions

$$T = \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \\ g_N^H \end{bmatrix}, \quad T x = \begin{bmatrix} g_1^H x \\ g_2^H x \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle x, g_1 \rangle \\ \langle x, g_2 \rangle \\ \vdots \end{bmatrix}$$

$N > M$

$$\hat{T}^* T = [\hat{g}_1 \ \ \hat{g}_2 \ \dots \ \hat{g}_N] \begin{bmatrix} \langle x, g_1 \rangle \\ \langle x, g_2 \rangle \\ \vdots \end{bmatrix} = \sum_{l=1}^{N} \langle x, g_l \rangle \hat{g}_l$$

Th. 1.23. $A : \mathcal{H} \to \ell^2$ a bounded linear operator.

$A^* A$ is invertible on $\mathcal{H}$ ($A^H A$ is inv.)

Then, the operator $A^+ = (A^* A)^{-1} A^*$ is a left-inv. of A.

Every solution of $LA = I_{\mathcal{H}}$ can be written as

$$L = A^+ + M(I_{\ell^2} - AA^+)$$

$$A^+ A = \underbrace{(A^* A)}^{-1} \underbrace{A^* A} = I$$

identify A with T, $A = T$

in finite dimensions $LT = I$

dual frame

$$[h_1 \; h_2 \ldots] \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix} x = \sum_{\ell=1}^{N} \langle x, g_\ell \rangle h_\ell$$

$$L = T^+ + M(I_{\ell^2} - TT^+)$$

$$\underline{\underline{T^{+}}} = (\underbrace{T^*T}_{S})^{-1} T^* = S^{-1} T^* = \underline{\underline{\tilde{T}^*}}$$

$$\tilde{T} = T S^{-1}$$

left-inversion with the pseudo-inverse amounts to synthesis with the minimal (or canonical) dual frame.

Th. 1.24. $\quad P = T S^{-1} T^*$

$$P : \ell^2 \to R(T) \subseteq \ell^2$$

has the following properties:

1. P is the identity operator on $R(T)$ (range space of T)
2. P is the zero operator on $R(T)^{\perp}$, where $R(T)^{\perp}$ is the orth. complement of $R(T)$.

Proof. 1. $c = Tx$

$$Pc = PTx = T S^{-1} \overbrace{T^*T}^{S} x = T \overbrace{S^{-1} S}^{I} x = Tx = c$$

$\Rightarrow P = I$ on $R(T)$.

2. $c \in R(T)^{\perp} \Rightarrow T^{*}c = 0$

$\langle d, Tx \rangle = 0$

$\langle d, Tx \rangle = \langle T^{*}d, x \rangle = 0, \forall x \in \mathcal{H}$

d is in $\mathcal{N}(T^{*})$

$$Pc = TS^{-1}\underbrace{T^{*}c}_{=0} = 0$$

$\boxed{P = TS^{-1}T^{*} \text{ is the orthogonal projection onto } R(T).}$

$\mathcal{N}(I-P)$

$L = T^{+} + M(I - TT^{+})$

$$TT^{+} = T\underbrace{S^{-1}T^{*}}_{T^{+}} = P$$

$$\boxed{\tilde{T} = TS^{-1}}$$

$$T^{+} = S^{-1}T^{*} = S^{-1}\underbrace{SS^{-1}}T^{*} = \overbrace{S^{-1}T^{*}}^{\tilde{T}^{*}}\underbrace{TS^{-1}T^{*}}_{P}$$

$$= \tilde{T}^{*}P$$

$$L = \tilde{T}^{*}\underset{R(T)}{P} + M(\underbrace{I - P})$$

orth. proj. op. onto orth. complement of $R(T)$

1.3.4 Tight frames

Parseval frames

Def. 1.25. A frame $\{g_k\}_{k\in K}$ with tightest possible frame bounds $A=B$ is called a tight frame.

$$A\|x\|^2 \leq \sum_{k\in K} |\langle x, g_k\rangle|^2 \leq B\|x\|^2$$

$$A = B \Rightarrow A\|x\|^2 = \sum_{k \in K} |\langle x, g_k \rangle|^2$$

ONB $\{e_k\}_{k \in \mathbb{N}}$ $\qquad \|x\|^2 = \sum_{k \in \mathbb{N}} |\langle x, e_k \rangle|^2$

$$\underbrace{\{ e_1, e_1, e_2, e_2, \dots \}}_{\{g_k\}_{k \in K}} \Rightarrow \sum_{k \in K} |\langle x, g_k \rangle|^2 = 2 \sum_{k \in \mathbb{N}} |\langle x, e_k \rangle|^2$$

$$= 2\|x\|^2$$

$$\underset{\substack{| \\ A=2}}{2}\|x\|^2 = \sum_{k \in K} |\langle x, g_k \rangle|^2$$

$$\left\{ e_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \dots \right\}$$

$$\|x\|^2 = \sum_{k \in K} |\langle x, g_k \rangle|^2$$

tight frame with $A = 1$.

Th. 1.26. $\{g_k\}_{k\in K}$ is a frame for $\mathcal{H}$. The frame $\{g_k\}_{k\in K}$ is tight with frame bound A iff $S = A I_{\mathcal{H}}$, or equivalently, if

$$x = \frac{1}{A}\sum_k \langle x, g_k\rangle g_k .$$

Proof. $$Sx = \sum_k \langle x, g_k\rangle g_k = A I_{\mathcal{H}} x$$

$$\Rightarrow x = \frac{1}{A}\sum_k \langle x, g_k\rangle g_k$$

$$\left(S = A I_H \Rightarrow S^{-1} = \frac{1}{A} I_H \Rightarrow \widetilde{g_k} = S^{-1} g_k = \frac{1}{A} g_k \right)$$

$$S - AI \geq 0$$

1. tightness of $\{g_k\} \Rightarrow S = A I_H$

$$AI \leq S \leq BI$$

$$\langle Sx, x\rangle = A\langle x, x\rangle$$

$$\left(A\|x\|^2 \leq \langle Sx, x\rangle \leq B\|x\|^2 \right)$$

$$\langle Sx, x\rangle = A\|x\|^2$$

$$\langle (S - AI_H)x, x\rangle = 0, \quad \forall x \in \mathcal{H}$$

$$S - A I_H = 0 \Rightarrow S = A I_H$$

2. $S = A I_H \Rightarrow \{g_k\}$ is tight

$$\Updownarrow$$

$$x = \frac{1}{A} \sum_k \langle x, g_k \rangle g_k$$

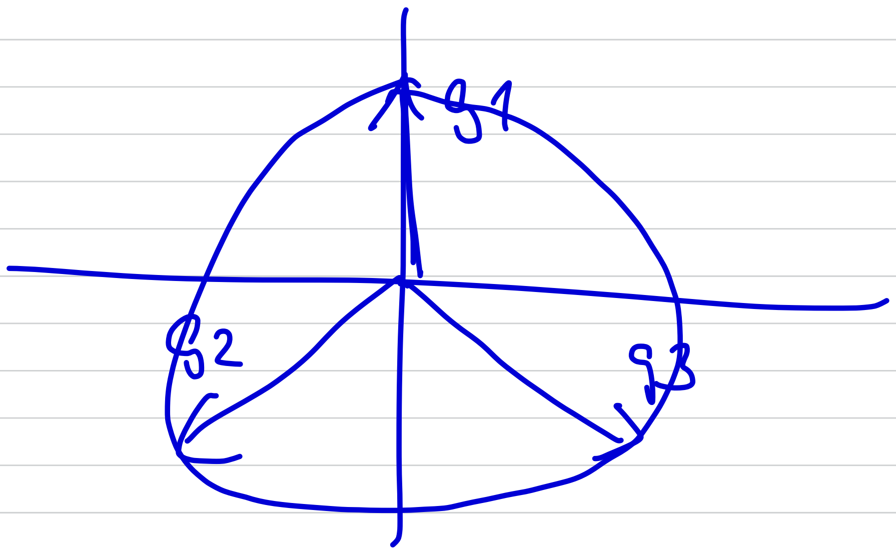
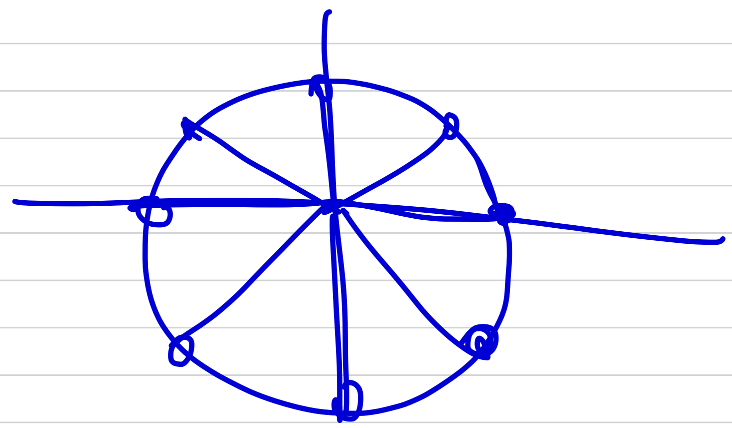
$$\langle x, x \rangle = \left\langle \frac{1}{A} \sum_k \langle x, g_k \rangle g_k , x \right\rangle =$$

$$= \frac{1}{A} \sum_k \langle x, g_k \rangle \overbrace{\langle g_k, x \rangle}^{\langle x, g_k \rangle^*} = \frac{1}{A} \sum_k |\langle x, g_k \rangle|^2$$

$$A\|x\|^2 = \sum_k |\langle x, g_k \rangle|^2 . \square$$

Ex. 1.27. $\mathbb{R}^2$

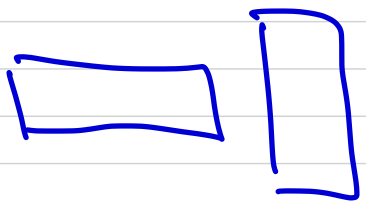
$$g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \quad g_3 = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

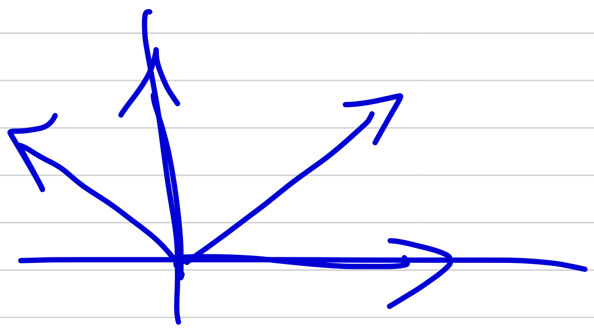
 

$$T = \begin{bmatrix} g_1 H \\ g_2 H \\ g_3 H \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$S = \frac{3}{2} I_2$$

$$S = T^H T = A I \qquad S = \frac{3}{2} I_2$$





Löwdin orthogonalization

Th. 1.28. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ with frame operator S. Denote the pos. def. square root of S^{-1} by $S^{-1/2}$. Then $\{S^{-1/2} g_k\}_{k\in K}$ is a tight frame for $\mathcal{H}$ with frame bound $A=1$, i.e.,

$$x = \sum_k \langle x, S^{-1/2} g_k \rangle S^{-1/2} g_k, \quad \forall x \in \mathcal{H}.$$

Proof.

$$S^{-1} S^{-1/2} = S^{-1/2} S^{-1} \quad \Big| \; S \cdot$$

$$\underbrace{S S^{-1}}_{I} S^{-1/2} = S S^{-1/2} S^{-1}$$

$$S^{-1/2} = S S^{-1/2} S^{-1} \quad \Big| \cdot S$$

$$S^{-1/2} S = S S^{-1/2}$$

$$x = I x = S^{-1} S x = S^{-1/2} \overbrace{S^{-1/2} S}^{S S^{-1/2}} x$$

$$= S^{-1/2} S \overbrace{S^{-1/2} x}^{y}$$

$$= S^{-1/2} S y = S^{-1/2} \sum_k \langle y, g_k \rangle g_k$$

$$= S^{-1/2} \sum_k \langle S^{-1/2} x, g_k \rangle g_k$$

$$= \sum_k \langle x, S^{-1/2} g_k \rangle S^{-1/2} g_k . \quad \square$$

$$S = \begin{pmatrix} x & & & \emptyset \\ & x & & \\ & & x & \\ \emptyset & & & x \end{pmatrix} \qquad d_1 \quad d_2 \quad \ddots$$

$$\frac{1}{\sqrt{|d_1|^2}} d_1$$

$$\frac{1}{\sqrt{|d_2|^2}} d_2$$

$$A = A^{1/2} A^{1/2}$$

$$A = \underbrace{\begin{bmatrix} b_{ij} \end{bmatrix}}_{N \times N} \qquad A^H A = I$$

$$T$$

$$\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \end{bmatrix}$$

$$T^H T = A I_H$$

Th. 1.29. A tight frame $\{g_k\}_{k\in K}$ with $A=1$ and $\|g_k\|=1$, $\forall k\in K$, is an ONB.

Proof.

$$\langle S g_k, g_k\rangle = A\|g_k\|^2 = \|g_k\|^2 = 1$$

$$\langle S g_k, g_k\rangle = \Big\langle \sum_j \langle g_k, g_j\rangle g_j, g_k \Big\rangle$$

$$= \sum_j \langle g_k, g_j\rangle \langle g_j, g_k\rangle$$

$$= \sum_j |\langle g_k, g_j\rangle|^2$$

$$= \underbrace{\|g_k\|^4}_{1} + \sum_{j\neq k} |\langle g_k, g_j\rangle|^2$$

$$= 1$$

$$\sum_{j \neq k} |\langle g_j, g_k \rangle|^2 = 0 \Rightarrow \langle g_j, g_k \rangle = 0, \forall j \neq k \quad \square$$

$$\left\{ e_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \dots \right\}$$

1.3.5 Exact frames and biorthonormality

Finite dimensions

$$x = \sum_{k=1}^{n} \langle x, e_k \rangle \widehat{e_k}$$

1. counting: no. of elements in $\{g_k\}$ is equal to dim. of ambient space.

2. For a given basis $\{e_k\}$, there is a unique basis $\{\widehat{e_k}\}$ s.t.

$$x = \sum_k \langle x, e_k \rangle \widehat{e_k} = \sum_k \langle x, \widehat{e_k} \rangle e_k$$

3. $\{e_k\}$ & $\{\tilde{e_k}\}$ are biorthonormal, i.e.,

$$\langle e_k, \tilde{e_\ell} \rangle = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

Def. 1.32. 1. $\{g_k\}_{k \in K}$ is <u>exact</u> if, for all $m \in K$, the set

$\{g_k\}_{k \neq m, k \in K}$ is incomplete for $\mathcal{H}$.

2. $\{g_k\}_{k \in K}$ is <u>inexact</u> if there is at least one element, g_m, so that $\{g_k\}_{k \neq m, k \in K}$ is again a frame for $\mathcal{H}$.

Lemma 1.33 $\{g_k\}$, $\{\tilde{g_k}\}$, $x \in \mathcal{H}$, $x = \sum_k \underbrace{\langle x, \tilde{g_k} \rangle}_{c_k} g_k$

if there are scalars $\{a_k\}_{k \in K}$ $\neq$ $\{\underline{c_k}\}_{k \in K}$ s.t.

$x = \sum_k a_k g_k$, then we must have

$$\boxed{\sum_k |a_k|^2 = \sum_k |c_k|^2 + \sum_{k \in K} |c_k - a_k|^2}$$

Proof.

$$c_k = \langle x, g_k \rangle = \langle x, S^{-1} g_k \rangle = \langle \underbrace{S^{-1} x}_{\tilde{x}}, g_k \rangle = \langle \tilde{x}, g_k \rangle$$

$$\langle x, \tilde{x} \rangle = \langle \sum_k c_k g_k, \tilde{x} \rangle = \sum_k c_k \underbrace{\langle g_k, \tilde{x} \rangle}_{c_k^*} = \sum_k |c_k|^2$$

$$\|$$

$$\langle x, \tilde{x} \rangle = \langle \sum_k a_k g_k, \tilde{x} \rangle = \sum_k a_k \underbrace{\langle g_k, \tilde{x} \rangle}_{c_k^*} = \sum_k a_k c_k^*$$

$$\sum_k |c_k|^2 + \sum_k |c_k - a_k|^2 = \sum_k |c_k|^2 + \sum_k (c_k - a_k)(c_k^* - a_k^*)$$

$$= \sum_k |c_k|^2 + \sum_k |c_k|^2 - \sum_k c_k a_k^*$$

$$- \sum_k a_k c_k^* + \sum_k |a_k|^2$$

$$= \sum_k |a_k|^2 . \quad \square$$

2nd ancillary result

Lemma 1.34 : $\{g_k\}$ & $\{\tilde{g_k}\}$

$$\sum_{k \neq m} |\langle g_m, \tilde{g_k}\rangle|^2 = \frac{1 - |\langle g_m, \tilde{g_m}\rangle|^2 - |1 - \langle g_m, \tilde{g_m}\rangle|^2}{2}$$

Proof. $g_m = \sum_{k \in K} a_k g_k$, $a_m = 1$, $a_k = 0, \forall k \neq m$

$g_m = \sum_{k \in K} c_k g_k$, $c_k = \langle g_m, \tilde{g_k}\rangle$

$$\sum_{k} |a_k|^2 = \sum_{k} |c_k|^2 + \sum_{k} |c_k - a_k|^2$$

$$1 = \sum_{k} |c_k|^2 + |c_m - a_m|^2 + \sum_{\substack{k \neq m \\ k \in K}} |c_k - a_k|^2$$

$$= \sum_{k} |\langle g_m, \widehat{g_k} \rangle|^2 + |\langle g_m, \widehat{g_m} \rangle - 1|^2$$

$$+ \sum_{k \neq m} |\langle g_m, \widehat{g_k} \rangle|^2$$

$$= |\langle g_m, \widehat{g_m} \rangle|^2 + 2 \sum_{k \neq m} |\langle g_m, \widehat{g_k} \rangle|^2 + |1 - \langle g_m, \widehat{g_m} \rangle|^2$$

$$\sum_{k \neq m} |\langle g_m, \widehat{g_k} \rangle|^2 = \frac{1 - |\langle g_m, \widehat{g_m} \rangle|^2 - |1 - \langle g_m, \widehat{g_m} \rangle|^2}{2} \qquad \square$$

Th. 1.35. $\{g_k\}$ & $\{\tilde{g_k}\}$

1. $\{g_k\}$ is exact iff $\langle g_m, \tilde{g_m} \rangle = 1, \ \forall m \in K$

2. $\{g_k\}$ is inexact iff there is at least one $m \in K$ s.t.
$\langle g_m, \tilde{g_m} \rangle \neq 1.$