Th. 1.35. $\{g_k\}$ & $\{\widetilde{g_k}\}$

ONB: $\langle g_m, \widetilde{g_m}\rangle = \langle g_m, g_m\rangle = \|g_m\|^2 = 1.$

1. $\{g_k\}$ is exact iff $\langle g_m, \widetilde{g_m}\rangle = 1, \forall m \in K$

2. $\{g_k\}$ is inexact iff there is at least one $m \in K$ s.t. $\langle g_m, \widetilde{g_m}\rangle \neq 1.$

Proof.

$$\sum_{k \neq m} |\langle g_m, \widetilde{g_k}\rangle|^2 = \frac{1 - |\langle g_m, \widetilde{g_m}\rangle|^2 - |1 - \langle g_m, \widetilde{g_m}\rangle|^2}{2}$$

$$\langle g_m, \widetilde{g_k}\rangle = 0, \forall k \neq m$$

$$\langle g_m, \widetilde{g_m}\rangle = 1 \Rightarrow \langle g_m, \widetilde{g_k}\rangle = 0, \forall k \neq m$$

exact, uniqueness of expansion, biorthonormality

Corr. 1.36. Let $\{g_k\}_{k \in K}$ be a frame for $\mathcal{H}$. If $\{g_k\}_{k \in K}$ is exact, then $\{g_k\}$ & $\{\hat{g_k}\}$ are biorthonormal, i.e.,

$$\langle g_m, \hat{g_k} \rangle = \begin{cases} 1, & k=m \\ 0, & \text{else} \end{cases}.$$

Conversely, if $\{g_k\}$ & $\{\hat{g_k}\}$ are biorthonormal, then $\{g_k\}$ is exact.

Proof. 1. $\{g_k\}$ exact $\Rightarrow$ biorthonormality

$$\Downarrow$$

$$\langle g_m, \hat{g_m} \rangle = 1 \quad \Longrightarrow \quad \langle g_m, \hat{g_k} \rangle = 0, \ \forall k \neq m$$

$$\langle g_m, \hat{g_k} \rangle = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases}$$

2. $\{g_k\}$ & $\{\hat{g_k}\}$ are biorthonormal $\Rightarrow$ $\{g_k\}$ is exact

$$\Downarrow$$

$$\langle g_k, \hat{g_m} \rangle = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases} \quad \Rightarrow \quad \underline{\langle g_m, \hat{g_m} \rangle = 1, \ \forall m \in K}$$

$$\Vert$$

$$\Downarrow$$

$\{g_k\}$ is exact. $\square$

Th. 1.37. If $\{g_k\}$ is an exact frame for $\mathcal{H}$ and $x = \sum_k c_k g_k$ with $x \in \mathcal{H}$, then the coefficients $\{c_k\}$ are unique and are given by

$$c_k = \langle x, \tilde{g}_k \rangle,$$

where $\{\tilde{g}_k\}$ is the canonical dual of $\{g_k\}$.

Proof.

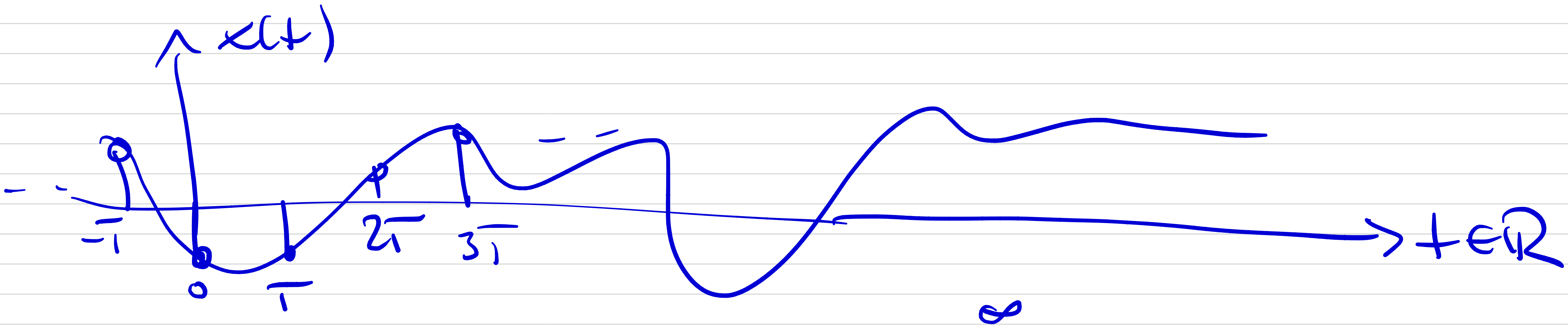
$$x = \sum_k c_k g_k \quad \Big| \ \langle \cdot, \tilde{g}_m \rangle$$

$$\{g_k\} \text{ exact} \iff \langle g_k, \tilde{g}_m \rangle = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

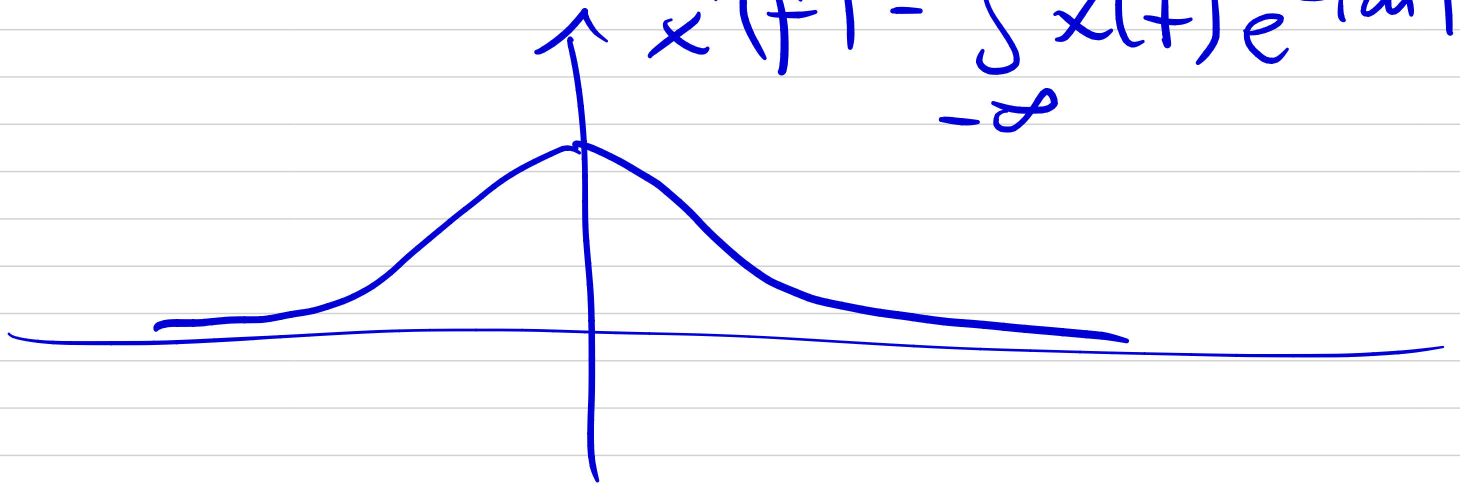
$$\langle x, \tilde{g}_m \rangle = \sum_k c_k \underbrace{\langle g_k, \tilde{g}_m \rangle}_{= \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}} = c_m$$

$\square$

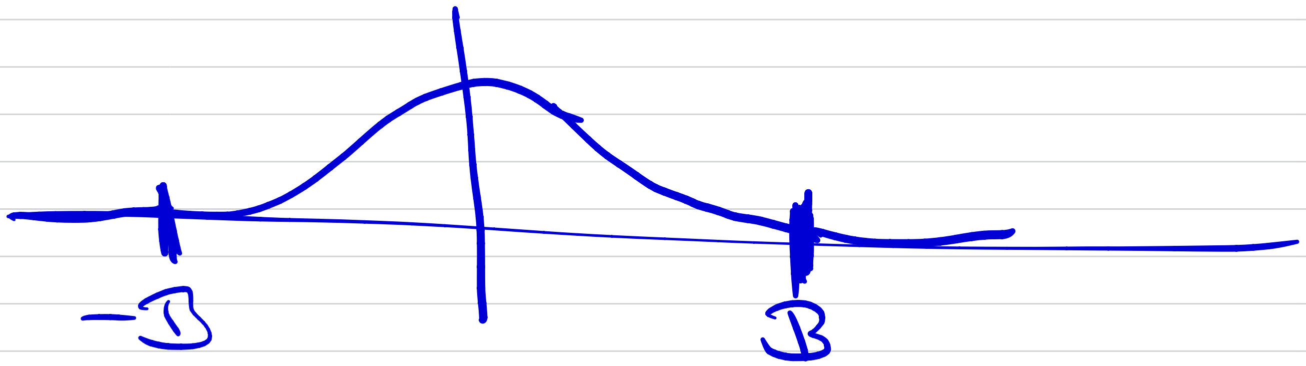
1.4. The sampling theorem



$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi f \cdot} \rangle$$



Bernstein: $|x'(t)| \leq c\, B$



$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft}\, dt \quad , \quad \hat{x}(f) = 0, \ |f| > B$$

$$x(t) = \int \hat{x}(f)\, e^{i2\pi ft}\, df$$

$$x[k] := x(kT)$$

Q: Can we reconstruct $x(t)$ from $x[k] = x(kT)$.

$$\hat{x}_d(f) := \sum_{k=-\infty}^{\infty} x[k]\, e^{-i2\pi kf} \qquad (\text{DTFT})$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\, e^{-i2\pi kf}$$

$$\overset{!}{=} \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{x}\left(\frac{f+k}{T}\right)$$

Fourier series

$$= \sum_{\ell} \overset{x(-\ell T)}{c_\ell}\, e^{i2\pi \ell f} = \sum_{\ell} x(\ell T)\, e^{-i2\pi \ell f}$$

$\frac{1}{T} \sum_{k} \hat{x}\left(\frac{f+k}{T}\right)$ is 1-periodic $\Rightarrow$ can expand it into a Fourier series

$$\frac{1}{T}\sum_k \hat{x}\left(\frac{f+1+k}{T}\right) = \frac{1}{T}\sum_{k'} \hat{x}\left(\frac{f+k'}{T}\right)$$

$k' = k+1$

$f'T - k = f$

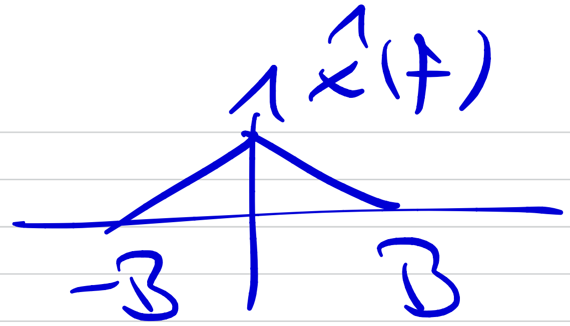
$$c_\ell = \int_0^1 \frac{1}{T}\sum_k \hat{x}\left(\frac{f+k}{T}\right) e^{-i2\pi f \ell}\, df \quad , \quad f' = \frac{f+k}{T}$$

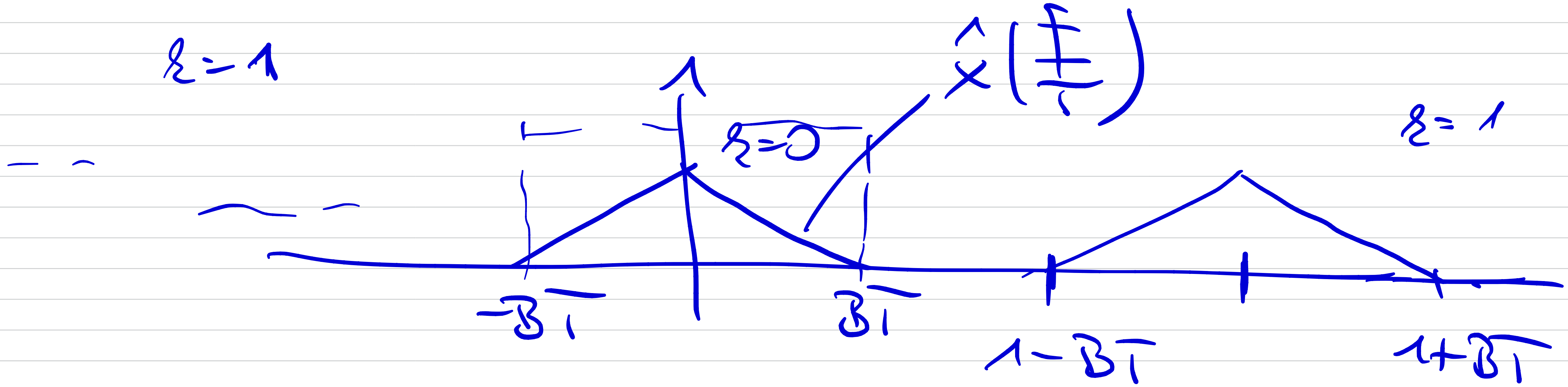
$$= \sum_k \frac{1}{\cancel{T}} \int_{\frac{k}{T}}^{\frac{k+1}{T}} \hat{x}(f')\, e^{-i2\pi \ell (f'T-k)}\, d(\cancel{f'T})$$

$$= \int_{-\infty}^{\infty} \hat{x}(f)\, e^{-i2\pi \ell f T}\, df = x(-\ell T)$$

$$\left(x(t) = \int \hat{x}(f)\, e^{i2\pi f t}\, df \right)$$

now study $\frac{1}{T}\sum_{k}\hat{x}\left(\frac{f+k}{T}\right)$ for





$k=0$: $\frac{1}{T}\hat{x}\left(\frac{f}{T}\right)$

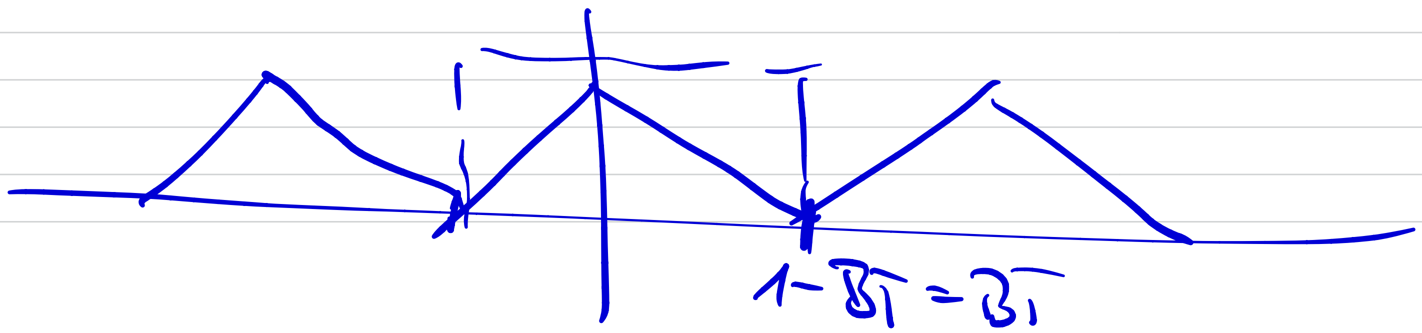
$k=-1$: $\frac{1}{T}\hat{x}\left(\frac{f-1}{T}\right)$,

$$\frac{f-1}{T} = \pm B$$

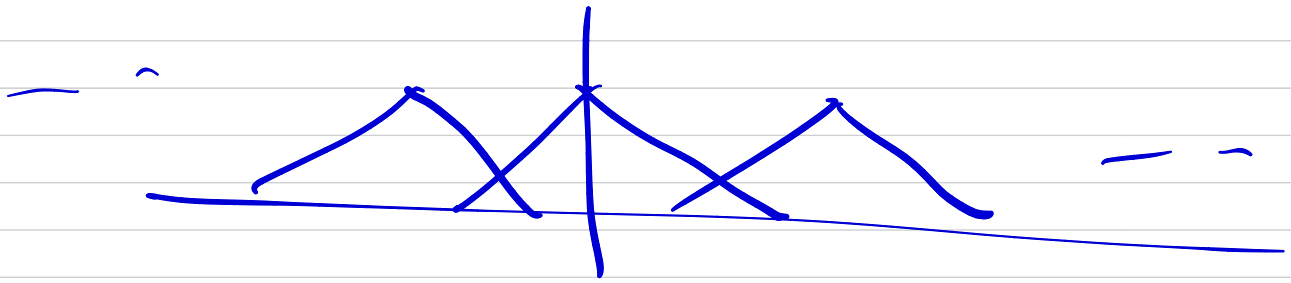
$$f = 1 \pm BT$$

1. ✓

2.



3.



$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\widehat{(x*h)}(f) = \hat{x}(f)\,\hat{h}(f)$$

$$1 - BT \geq BT$$

$$1 \geq 2BT$$

$$\underbrace{\frac{1}{T}} \geq 2B$$

$$\boxed{f_s \geq 2B}$$

$$\hat{h}_{LP}(f) = \begin{cases} 1, & |f| \leq BT \\ 0, & \text{else} \end{cases}$$

$$\hat{x}\left(\frac{F}{T}\right) = \underbrace{\hat{x_d}(f)}_{\sum_k x(kT)\, e^{-i2\pi k F}} \, T \, \hat{h}_{LP}(f)$$

obtained from samples

$$\hat{x}(f) = \hat{x_d}(fT)\, T\, \hat{h}_{LP}(fT)$$

$$x(t) = \int \hat{x_d}(fT)\, T\, \hat{h}_{LP}(fT)\, e^{i2\pi f t}\, df$$

$$= \int_{-B}^{B} \sum_k x(kT)\, e^{-i2\pi k f T}\, T\, e^{i2\pi f t}\, df$$

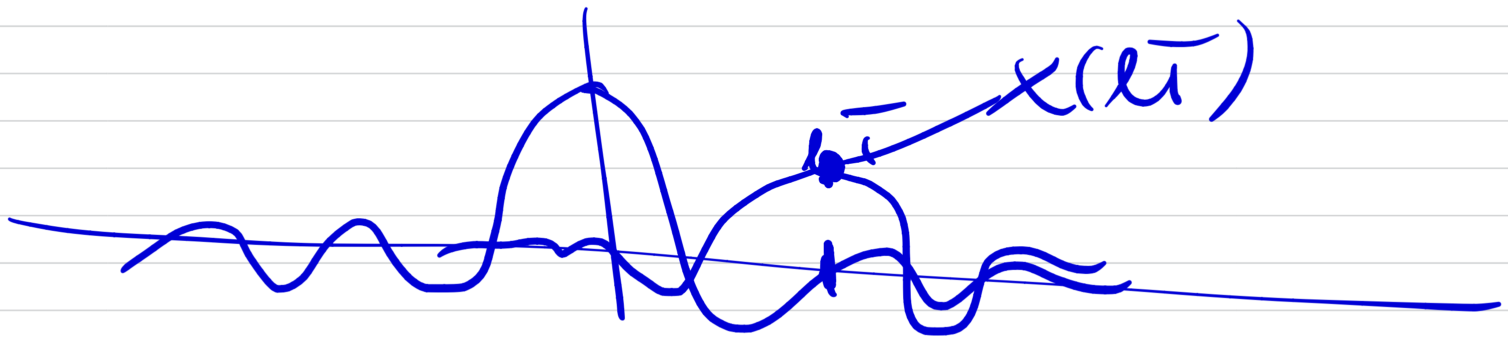
$$= \sum_{k} x(kT) \int_{-B}^{B} 1 \cdot e^{i2\pi f T\left(\frac{t}{T} - k\right)} df$$

$$\overset{\text{F.T. table}}{=} 2BT \sum_{k} x(kT) \operatorname{sinc}(2B(t-kT))$$

$$\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$

Th. 1.38 (Sampling theorem): Let $x \in \mathcal{L}^2(B)$. Then, $x(t)$ is uniquely specified by its samples $x(kT)$, $k \in \mathbb{Z}$, if $\frac{1}{T} \geq 2B$. Specifically, we can reconstruct $x(t)$ from $x(kT)$, $k \in \mathbb{Z}$, according to

$$x(t) = 2BT \sum_{k} x(kT) \operatorname{sinc}(2B(t-kT)).$$



1.4.1. Sampling theorem as a frame expansion

$$g_k(t) = 2B \operatorname{sinc}(2B(t-kT)), \quad k \in \mathbb{Z}$$

$$x(kT) \overset{\checkmark}{=} \int_{-B}^{B} \hat{x}(f)\, e^{i2\pi kTf}\, df \overset{!}{=} \langle \hat{x}, \hat{g}_k \rangle \overset{\text{Plancherel}}{=} \langle x, g_k \rangle$$

$$\left(x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\, e^{i2\pi tf}\, df \right)$$

$$\underline{g_k(t) = 2B \operatorname{sinc}(2B(t-kT))} \quad \circ\!\!-\!\!\bullet \quad \begin{cases} e^{-i2\pi kTf}, & |f| \le B \\ 0, & \text{else} \end{cases}$$

$$\langle \hat{x}, \hat{g_k} \rangle = \int_{-\cancel{\sigma}\,-B}^{\cancel{\sigma}\,B} \hat{x}(f)\, e^{i2\pi k T f}\, df = x(kT)$$

$$x(t) = T \sum_k \langle x, g_k \rangle g_k$$

$$\|x\|^2 = \langle x, x \rangle = \left\langle T \sum_k \langle x, g_k \rangle g_k, x \right\rangle$$

$$= T \sum_k |\langle x, g_k \rangle|^2$$

$$\frac{1}{T}\|x\|^2 = \sum_k |\langle x, g_k \rangle|^2 , \quad A = B = \frac{1}{T}$$

$$A\|x\|^2 \leq \sum_k |\langle x, g_k \rangle|^2 \leq B\|x\|^2$$

Tight frame expansion with $\underline{\underline{A = B = \frac{1}{\tau}}}$

$$T : x \rightarrow \{\langle x, g_k \rangle\}_{k \in \mathbb{Z}}$$ sampling operator

$$T^* : \{c_k\} \rightarrow \sum_k c_k g_k$$ interpolation operator

$$S = T^* T$$

$$Sx = \sum_k \langle x, g_k \rangle g_k$$

$$\frac{1}{\tau} \|x\|^2 = \sum_k |\langle x, g_k \rangle|^2 \quad , A = B = \frac{1}{\tau}$$

$$\tilde{g_k} = T g_k$$

$$\langle g_m, g_{\tilde m} \rangle \overset{?}{=} 1$$

$$\langle g_m, T g_m \rangle = T \langle g_m, g_m \rangle = T \|g_m\|^2 = T \|\hat{g}_m\|^2 = 2BT$$

$$\hat{g}_m(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{else} \end{cases}$$

$\langle g_m, g_{\tilde m} \rangle = 2BT = 1$ for critical sampling, i.e., $\frac{1}{T} = 2B$

$$g_k'(t) = \sqrt{T}\, g_k(t)$$

$$x(t) = \sum_k \langle x, g_k' \rangle g_k' \Rightarrow \left(x(t) = \frac{1}{A} \sum_k \langle x, g_k \rangle g_k \right)$$

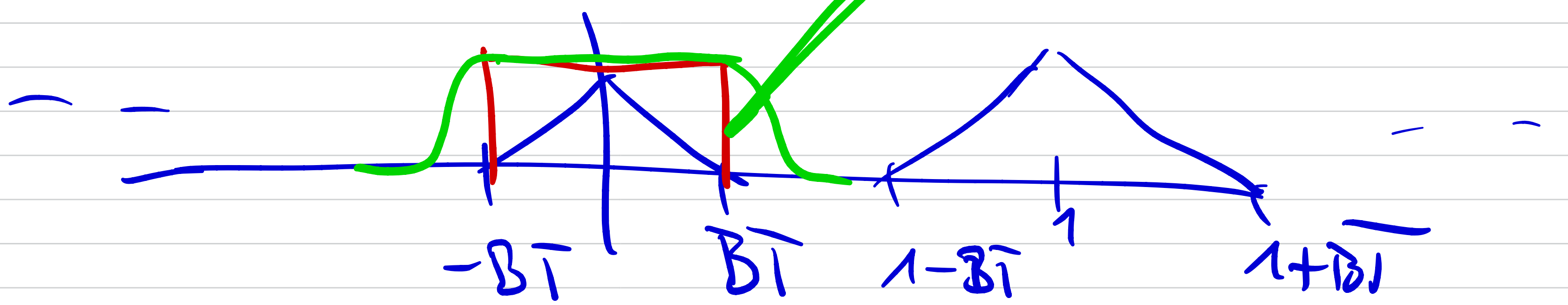
$$\|g_k'\|^2 = T \|g_k\|^2 = 2BT = 1$$

$\{g_k'\}$ is tight, with $A = 1$ and $\|g_k'\| = 1 \Rightarrow \{g_k\}$ is an ONB ✓

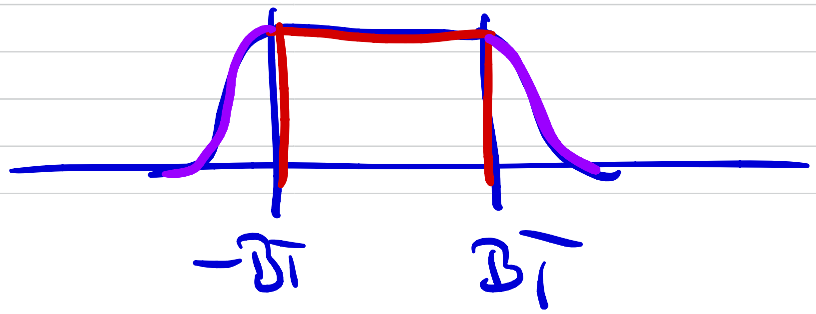
14.2. Design freedom in oversampled AD conversion

$$x(t) = \sum_k x(kT) \underbrace{\int_{-\infty}^{\infty} \hat{h}_{LP}(fT)\, e^{i2\pi fT(t/T - k)}\, d(fT)}$$

$$= \sum_k x(kT)\, h_{LP}\left(\frac{t}{T} - k\right)$$



$$x(t) = \sum_k x(kT)\, h\left(\frac{t}{T} - k\right)$$

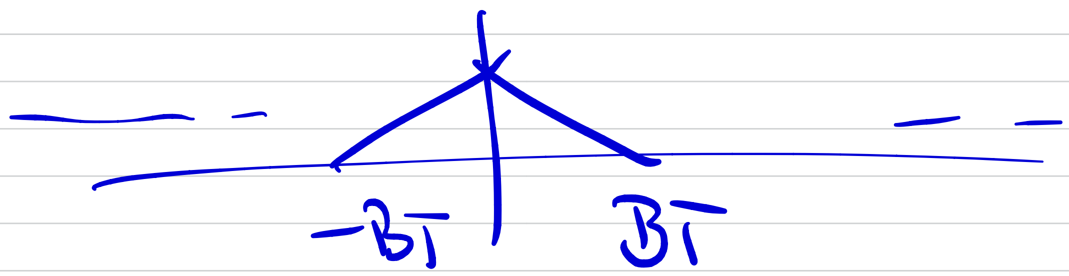


$$L = \hat{T}^* P + \mu(I_{\ell^2} - P)$$

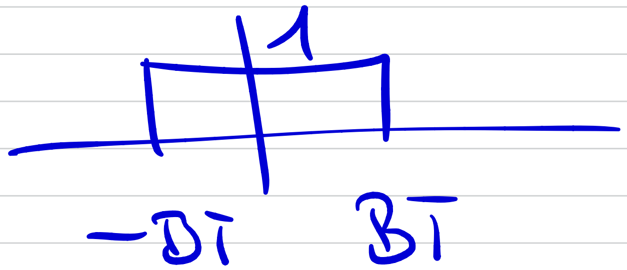
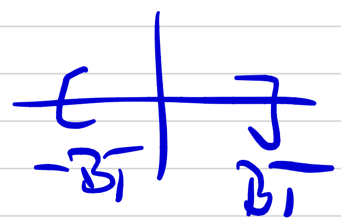
P ... orth. proj. onto $R(T)$

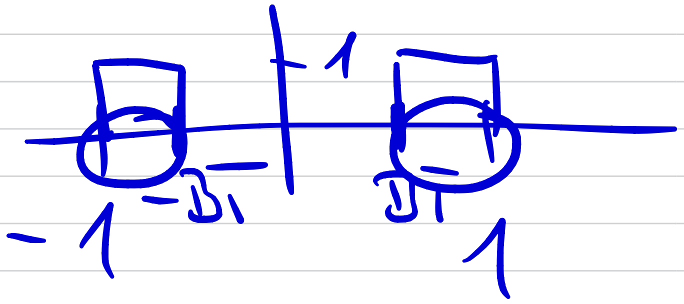
$T : x \to \{\langle x, g_k \rangle = x(kT)\}$

$$\sum_k x(kT)\, e^{-i 2\pi k T f} = \frac{1}{T} \sum_k \hat{x}\left(\frac{f+k}{T}\right)$$



$R(T)$ is the space of sequences with DTFT supported on $[-BT, BT]$

 $= P$ is the orth. proj. op. onto 



$$= I - P$$

$$\hat{x}(f) = \sum_k x(k) e^{-i 2\pi k f}$$

$$L = \hat{T}^* P + \mu (I_{\ell_2} - P)$$

$$h(t) = h_{LP}(t) + h_{out}(t)$$

$$\hat{h}_{LP}(f) = \begin{cases} 1, & |f| \leq B \\ 0, & \text{else} \end{cases}$$

$$\hat{h}_{out}(f) = \begin{cases} \text{arb}(f), & BT \leq |f| \leq 1/2 \\ 0, & \text{else} \end{cases}$$

$$A : \{c_k\}_{k \in \mathbb{Z}} \to \sum_k c_k \, h_{LP}\left(\frac{t}{T} - k\right)$$

$$B : \{c_k\}_{k \in \mathbb{Z}} \to \sum_k c_k \, h_{out}\left(\frac{t}{T} - k\right)$$

$$\int \sum_k c_k \, h_{LP}\left(\frac{t}{T} - k\right) e^{-i2\pi f t} \, dt$$

$$= \sum_k c_k \int h_{LP}\left(\frac{t}{T} - k\right) e^{-i2\pi f t} \, dt$$

$$t' = \frac{t}{T} - k \quad \Rightarrow (t' + k) T = t$$

$$= \sum_k c_k T \int h_{LP}(t') \, e^{-i2\pi f (t' + k) T} \, dt'$$

$$= \underbrace{\sum_k c_k e^{-i2\pi f k T}}_{\hat{C}(fT)} \underbrace{\int h_{LP}(t) e^{-i2\pi f T t} \, dt}_{\hat{h}_{LP}(fT)}$$

$$x(t) = \sum_k x(kT) \, h\left(\frac{t}{T} - k\right)$$

$$= \sum_{k} x(kT)\, h_{LP}\left(\frac{t}{T} - k\right) + \sum_{k} x(kT)\, h_{out}\left(\frac{t}{T} - k\right)$$