$$= \sum_{k} x(kT)\, h_{LP}\left(\frac{t}{T}-k\right) + \sum_{k} x(kT)\, h_{out}\left(\frac{t}{T}-k\right)$$

2. Uncertainty relations and signal recovery

Uncertainty relation (physics), position & momentum

$$\sigma_x \sigma_y \geq \frac{h}{2\pi}$$

in signal theory, $x(t) \circ\!\!-\!\!\bullet\ \hat{x}(f)$

$$x(at) \circ\!\!-\!\!\bullet\ \frac{1}{|a|}\hat{x}\left(\frac{f}{a}\right)$$

$$\sigma_x \sigma_{\hat{x}} \geq \text{const.} \implies \sigma_{\hat{x}} \geq \frac{\text{const.}}{\sigma_x}$$

σ_x ... eff. time duration $\sqrt{\int t^2 |x(t)|^2 dt}$

$\sigma_{\hat{x}}$ ... eff. bandwidth $\sqrt{\int f^2 |\hat{x}(f)|^2 df}$

$$\hat{x}(f) = \int x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi f \cdot} \rangle \quad \text{... analysis}$$

$$x(t) = \int \hat{x}(f)\, e^{i2\pi f t}\, df = \int \langle x(\cdot), e^{i2\pi f \cdot} \rangle \underbrace{e^{i2\pi f t}}_{\text{synthesis}}\, df$$

$$[F_m]_{k,l} = \frac{1}{\sqrt{m}} e^{i2\pi \frac{kl}{m}} \qquad \{ e^{i2\pi f t} \}_{f \in \mathbb{R}}$$

$$F_m = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{i2\pi/m} & e^{i2\pi 2/m} & \\ \vdots & \vdots & & \\ 1 & & & \end{bmatrix}, \quad k, l = 0, 1, \ldots, m-1$$

$$\hat{x} = F_m x \qquad F_m F_m^H = F_m^H F_m = I_m$$

$$x = F_m^H \hat{x}$$

$\Big(\quad \delta(t), \quad x(t) = \int \underbrace{x(t')\,\delta(t-t')}\, dt' = \int x(t)\,\delta(t-t')\,dt'$

$= x(t) \underbrace{\int \delta(t-t')\,dt'}_{=1}$

$= x(t)$ ✓

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[m-1] \end{bmatrix}, \qquad x[n] = \overbrace{\langle x[\cdot], \delta[\cdot - n] \rangle}^{\text{analysis}}$$

Kronecker delta function

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$$

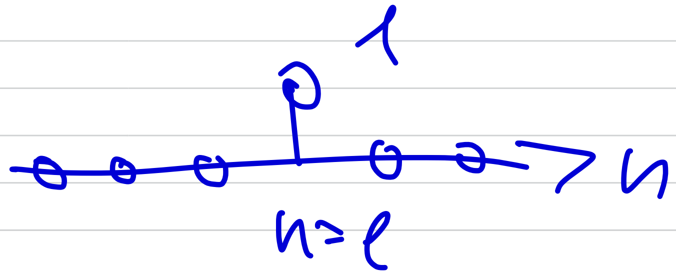
$$= \sum_{l=0}^{m-1} x[l]\,\delta[l-n] = x[n]$$

$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1]$$

$$+ \dots + x[m-1]\,\delta[n-(m-1)]$$

$$= \underbrace{\sum_{l=0}^{m-1} x[l]\,\delta[n-l]}_{\text{synthesis}}$$

$\delta[n-l] =$



$$= \sum_{l=0}^{m-1} \langle x[\cdot], \delta[\cdot - l] \rangle\, \delta[n-l]$$

$$x(t') = \int x(t)\,\delta(t-t')\,dt' = \langle x(\cdot), \delta(\cdot - t') \rangle$$

$$x(t) = \underbrace{\int \langle x(\cdot'), \delta(\cdot' - t) \rangle\, \delta(t-t')\,dt'}_{\text{analysis–synthesis}}$$

in finite dimensions (time-basis)

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & & \emptyset \\ \vdots & 0 & 1 & & \\ 0 & \emptyset & \vdots & & \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ e_1 & e_2 \end{matrix}$$

$$\langle x, e_1 \rangle = x[0]$$

$$\langle x, e_2 \rangle = x[1]$$

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{x} = \frac{1}{\sqrt{n}} \left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & \left[e^{i \frac{2\pi k \ell}{n}}\right] & \\ \vdots & & & \\ 1 & & & \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Applications of uncertainty relations

- quantum mechanics
- compressed sensing ←
- Gabor & Wilson frame theory
  (Balian-Low theorem)
- wireless communications (pulse shape design
  for OFDM)
- algebraic coding theory (BCH, RS)
- theory of PDE (smoothness properties of solutions,
  C. Fefferman)

Notation: $\mathcal{A} \subseteq \{1, \dots, m\}$, $D_{\mathcal{A}}$ is diagonal

$$(D_{\mathcal{A}})_{i,i} = \begin{cases} 1, & \text{if } i \in \mathcal{A} \\ 0, & \text{else} \end{cases}$$

$\mathcal{A} = \{1, 3, 5\}$, $m = 5$

$$D_{\mathcal{A}} = \begin{pmatrix} 1 & & & & \emptyset \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ \emptyset & & & & 1 \end{pmatrix}$$

$U \in \mathbb{C}^{m \times m}$ ... unitary matrix

$$P_{\mathcal{A}}(U) = U D_{\mathcal{A}} U^H = [u_1 \; u_2 \dots u_m] \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} u_1^H \\ u_2^H \\ \vdots \\ \vdots \end{bmatrix}$$

$$= \sum_{l \in \mathcal{A}} u_l u_l^H$$

$$P_{\mathcal{A}}(U)\, x = \left(\sum_{l \in \mathcal{A}} u_l u_l^H \right) x = \sum_{l \in \mathcal{A}} \langle x, u_l \rangle u_l$$

$P_{\mathcal{A}}(U)$ is an orth. projection operator

1. $P_{\mathcal{A}}(U)\, P_{\mathcal{A}}(U) = U D_{\mathcal{A}} \underbrace{U^H U}_{I} D_{\mathcal{A}} U^H = U \underbrace{D_{\mathcal{A}}^2}_{D_{\mathcal{A}}} U^H$

$$= U D_{\mathcal{A}} U^H = P_{\mathcal{A}}(U)$$

idempotent

2. $W^{U, \mathcal{A}} = R(P_{\mathcal{A}}(U))$

$$x = U D_A c = [u_1 \, u_2 \dots u_m] \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \end{bmatrix} c$$

$$= \sum_{l \in A} u_l c_l$$

$$P_A(U)x = U \overbrace{D_A \overbrace{U^H U}^{I} D_A}^{D_A} c = U D_A c = x, \qquad \underbrace{U D_A U^H}_{P_A(U)}$$

$$\Rightarrow \quad P_A(U) = I \text{ on } R(P_A(U))$$

$$\Rightarrow \quad P_A(U) \text{ is the orth. proj. op. onto } W^{U,A}$$

more notation: $x_A = D_A x$

$$|||A|||_2 = \max_{\underline{x : \|x\|_2 = 1}} \|Ax\|_2$$

$$\|A\|_2 = \sqrt{\mathrm{Tr}(AA^H)} \quad \text{-- Frobenius norm}$$

2.2 Uncertainty relations in $(\mathbb{C}^n, \|\cdot\|_2)$

D_P -- "time-limitation" operator

$$D_P x = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} x = x_P$$

$$\underbrace{\mathcal{F} D_P \mathcal{F}}_{P_P(\mathcal{F})} x$$

$$P_P(\mathcal{F}) x$$

$\uparrow$ poly

replace DFT-matrix F_m by $U \in \mathbb{C}^{m \times m}$

support set Q

$P_Q(U)$, $W^{Q,U} \leftarrow W^{U,Q}$

$$\boxed{\Delta_{P,Q}(U) = ||| D_P P_Q(U) |||_2}$$

$$||| P_P(A) P_Q(B) |||_2 = ||| A D_P A^H B D_Q B^H |||_2$$

A, B ... unitary

A^H , A

$$= ||| D_P (A^H B) D_Q (B^H A) |||_2$$

$$= ||| D_P P_Q(\overbrace{A^H B}^{U}) |||_2$$

$$\boxed{\Delta_{P,Q}(U) = \max_{x \in W^{U,Q} \setminus \{0\}} \frac{\|x_P\|_2}{\|x\|_2}}$$

Lemma 2.20. $U \in \mathbb{C}^{m \times m}$ unitary, $P, Q \subseteq \{1, \dots, m\}$,

$P_Q(U) = U D_Q U^H$ is the orth. proj. onto $W^{U,Q}$, Then

1. $|||P_Q(U) D_P|||_2 = |||D_P P_Q(U)|||_2$

2. $|||D_P P_Q(U)|||_2 = \max_{\substack{x \in W^{U,Q} \\ x \neq 0}} \frac{\|x_P\|_2}{\|x\|_2}$.

Proof. 1. $|||D_P P_Q(U)|||_2 = |||(D_P P_Q(U))^*|||_2$

$$= \left|\left|\left| P_Q^*(U)\, D_P^* \right|\right|\right|_2 = \left|\left|\left| P_Q(U) D_P \right|\right|\right|_2$$

2. $$\left|\left|\left| D_P\, P_Q(U) \right|\right|\right|_2 \overset{\text{Def.}}{=} \max_{x:\, \|x\|_2 = 1} \| D_P P_Q(U) x \|_2$$

$$= \max_{\substack{x:\, P_Q(U)x \neq 0 \\ \|x\|_2 = 1}} \| D_P P_Q(U) x \|_2$$

$$\leq \max_{x:\, P_Q(U)x \neq 0} \left\| D_P \frac{P_Q(U)x}{\|P_Q(U)x\|_2} \right\|_2$$

$$\left(\|P_Q(U)x\|_2 \leq \|x\|_2 = 1 \right)$$

$$= \max_{x \in W^{U,Q}} \frac{\|x_P\|_2}{\|x\|_2}$$

$$= \max_{x:\, P_e(U)x \neq 0} \left\| D_p P_e(U) \frac{P_e(U)\, x}{\|P_e(U)x\|_2} \right\|_2$$

$$\leq \max_{x:\, \|x\|_2 = 1} \|D_p P_e(U) x\|_2$$

$$= |||D_p P_e(U)|||_2 \quad . \square$$

Lemma 2.21. Let $A \in \mathbb{C}^{m\times n}$. Then,

$$\frac{\|A\|_2}{\sqrt{\operatorname{rank}(A)}} \leq \overbrace{|||A|||_2}^{\sigma_1} \leq \|A\|_2$$

Proof. $A \neq 0$, $r = \operatorname{rank}(A)$, $\overset{\text{largest}}{\sigma_1}, \dots, \overset{\text{smallest}}{\sigma_r} = \sigma(A)$ organized in decreasing order. $|||A|||_2 = \sigma_1$

$$\|A\|_2 = \sqrt{\sum_{i=1}^{r} \sigma_i^2}$$

$$\sigma_1 \leq \sqrt{\sum_{i=1}^{r} \sigma_i^2} \leq \sqrt{r}\, \sigma_1$$

$$\|A\|_2 \leq \sqrt{r}\, \sigma_1$$

$$\sqrt{\sum_{i=1}^{r} \sigma_i^2} \leq \sqrt{r}\, \sigma_1 .$$

$$\Delta_{P,a}(U) = \max_{x \in \mathbb{R}^{n,a} \setminus \{0\}} \frac{\|xp\|_2}{\|x\|_2} .$$

$$\Delta_{P,a}(U) \leq c$$ -- uncertainty relation

used

Lemma 2.21. $$\frac{\|D_P P_Q(U)\|_2}{\sqrt{\operatorname{rank}(D_P P_Q(U))}} \leq \Delta_{P,Q}(U) \leq \|D_P P_Q(U)\|_2$$

$$- \quad \|D_P P_Q(U)\|_2 = \sqrt{\operatorname{Tr}\big(D_P \underbrace{P_Q(U) P_Q^H(U)} D_P^H\big)}$$

$$= \sqrt{\operatorname{Tr}(D_P P_Q(U))}$$

$$- \quad \operatorname{rank}(D_P P_Q(U)) = \operatorname{rank}(D_P U D_Q U^H)$$

$$\leq \min(|P|, |Q|)$$

$\operatorname{rank}(AB) \leq \min(r(A), r(B))$

$$\sqrt{\frac{\operatorname{Tr}(D_P P_Q(U))}{\min(|P|, |Q|)}} \leq \Delta_{P,Q}(U) \leq \sqrt{\operatorname{Tr}(D_P P_Q(U))}$$

$U = F_m$

$$\sqrt{\operatorname{Tr}(D_P P_Q(F))} = \sqrt{\operatorname{Tr}(D_P F D_Q F^H)}$$

$$= \sqrt{\operatorname{Tr}(D_P F D_Q D_Q F^H D_P)}$$

$$= \sqrt{\sum_{i \in P} \sum_{j \in Q} |F_{i,j}|^2}$$

$$= \sqrt{\frac{|P|\,|Q|}{m}}$$

$$\sqrt{\frac{|P|\,|Q|}{\min(|P|,|Q|)\, m}} \leq \Delta_{P,Q}(U) \leq \sqrt{\frac{|P|\,|Q|}{m}}$$

$$\sqrt{\frac{\max(|P|,|Q|)}{m}} \leq \Delta_{P,Q}(u) \leq \sqrt{\frac{|P||Q|}{m}}$$

$P = \{1\}$, $Q = \{1, \dots, m\}$

$$\sqrt{\frac{m}{m}} \leq \Delta_{P,Q}(u) \leq \sqrt{\frac{m}{m}} = 1$$

$$\Rightarrow \underline{\underline{\Delta_{P,Q}(u) = 1}}$$

$P = \{\frac{m}{n}, \frac{2m}{n}, \dots, \frac{(n-1)m}{n}, m\}$, $|P| = n$

$m/n \in \mathbb{N}$

$Q = \{l+1, \dots, l+n\}$ interpreted circularly in $\{1, \dots, m\}$

$|Q| = n$

$$\boxed{\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(u) \leq \frac{n}{\sqrt{m}}}$$

Next week: - lower bound is tight

- upper bound can be improved to scale exactly like lower bound through large sieve.