$$\boxed{\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(\mathcal{U}) \leq \frac{n}{\sqrt{m}}}$$

Next week: - lower bound is tight ⟵
- upper bound can be ~~...~~ improved to scale exactly like lower bound through large sieve.

Lemma 2.1. Let n divide m and consider

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \dots, \frac{(n-1)m}{n}, m \right\}$$

$$Q = \{ \ell+1, \dots, \ell+n \},$$

where Q is interpreted circularly on $\{1, \dots, m\}$. Then,

$$\Delta_{P,Q}(\mathcal{F}) = \sqrt{\frac{n}{m}}.$$

Proof. We have

$$\Delta_{P,Q}(F) = |||D_P P_Q(F)|||_2$$

$$= |||P_Q(F) D_P|||_2$$

$$P_Q(F) = U D_Q U^H$$

$$P_Q(F)x = U D_Q U^H x = \sum_{\ell \in Q} u_\ell \langle x, u_\ell \rangle$$

$$= \max_{x : \|x\|_2 = 1} \| \cancel{U} D_Q F^H D_P x \|_2$$

$$= \max_{x : \|x\|_2 = 1} \| D_Q F^H \overbrace{D_P x}^{x_P} \|_2$$

$$= \max_{x \neq 0} \left\| D_Q F^H \frac{x_P}{\|x\|_2} \right\|$$

$$\sqrt{\frac{n}{m} \cancel{\|x\|_2}^2}$$

$$= \max_{\substack{x \neq 0 \\ x = x_P}} \frac{\| D_Q F^H x \|_2}{\cancel{\|x\|_2}} \qquad (2.21)$$

$$\|D \circ F^H x\|_2^2 = \frac{1}{m} \sum_{q \in Q} \Big| \sum_{p \in P} x_p \, e^{i 2\pi \frac{pq}{m}} \Big|^2 \qquad \Big(y_s = x_{\frac{ms}{n}} \Big)$$

$m \times m$ DFT matrix

$$= \frac{1}{m} \sum_{q \in Q} \Big| \sum_{s=1}^{n} x_{\frac{ms}{n}} \, e^{i 2\pi \frac{sq}{n}} \Big|^2 \qquad \Big(p = \frac{m}{n} s$$

$$= \frac{1}{m} \sum_{q \in Q} \Big| \sum_{s=1}^{n} y_s \, e^{i 2\pi \frac{sq}{n}} \Big|^2$$

$$= \frac{n}{m} \|F^H y\|_2^2$$

$n \times n$ DFT matrix

$$= \frac{n}{m} \|y\|_2^2 = \frac{n}{m} \|x\|_2^2$$

$$\Rightarrow \Delta_{P,Q}(F) = \sqrt{\frac{n}{m}} \, . \quad \text{q.e.d.}$$

$$\Delta_{P,Q}(F) \leq \sqrt{2}\sqrt{n/m}$$

$$\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(F) \leq \sqrt{2}\sqrt{n/m}$$

2.2.2. Coherence-based uncertainty relation

How do we get $\underbrace{\text{good}}_{\text{simple}}$ upper bounds on $\Delta_{P,Q}(U)$?

Def. 2.3. For $A = (a_1 \dots a_n) \in \mathbb{C}^{m \times n}$ with columns $\|\cdot\|_2$-normalized to 1, the coherence is defined as

$$\mu(A) = \max_{i \neq j} |a_i^H a_j| \left(= \max_{i \neq j} |\langle a_j, a_i \rangle| \right)$$

Lemma 2.4. Let $U \in \mathbb{C}^{m \times m}$ be unitary and $P, Q \subseteq \{1, \dots, m\}$. Then, we have

$$|P||Q| \geq \frac{\Delta_{P,Q}(U)}{\mu([I\ U])}$$

$$\Delta_{P,Q}(U) \leq \sqrt{|P||Q|}\,\mu([I\ U]).$$

Proof. $\Delta_{P,Q}^2(\mathcal{U}) \leq \operatorname{tr}(D_P U D_Q U^H)$

$= \operatorname{tr}(D_P D_P U D_Q D_Q U^H)$

$= \operatorname{tr}(D_P U D_Q \mid D_Q U^H D_P)$

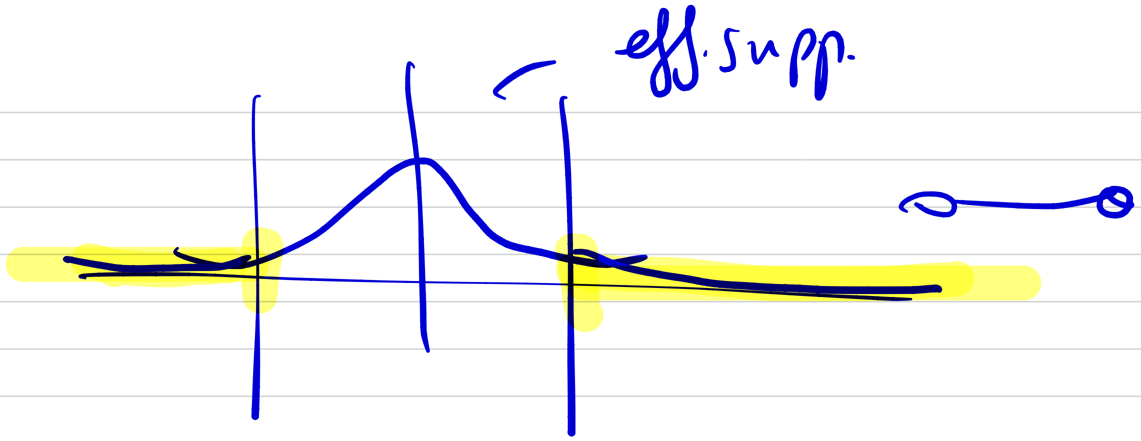
$$= \sum_{k \in P} \sum_{l \in Q} |u_{k,l}|^2$$

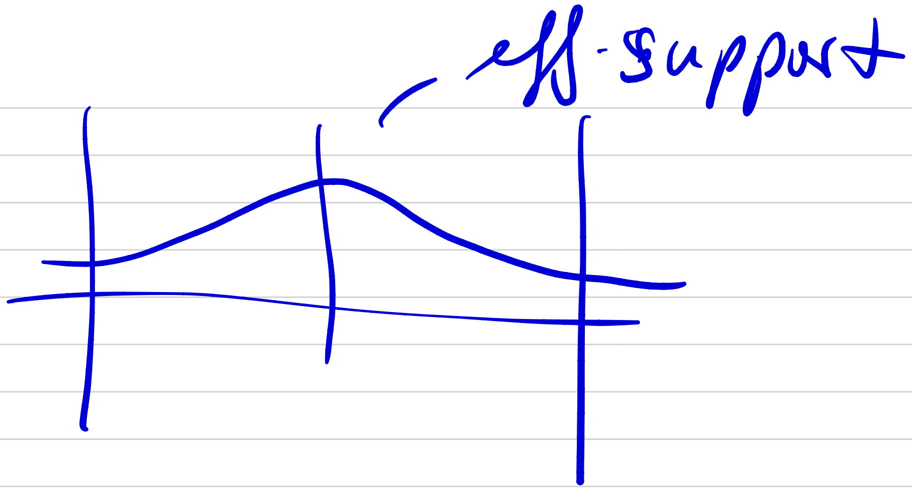
$$\leq |P||Q| \max_{k,l} |u_{k,l}|^2$$

$= |P||Q| \mu^2([I \; U])$. □

$\mu([I \; U]) = \mu([e_1 e_2 \ldots e_n \; u_1 u_2 \ldots u_n])$

2.2.3. Concentration Inequalities

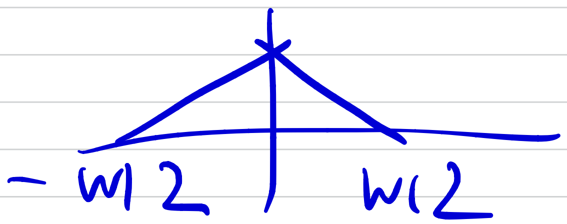




Def. 2.5. Let $P \subseteq \{1,\dots,m\}$ and $\varepsilon_P \in [0,1]$. The vector $x \in \mathbb{C}^m$ is said to be ε_P-concentrated if

$$\|x - x_P\|_2 \le \varepsilon_P \|x\|_2 .$$

$$\frac{\|x - x_P\|_2}{\|x\|_2} \le \varepsilon_P$$



$$\frac{1}{T_S} = f_S \ge 2\frac{W}{2} = W$$

take a sample every $T_S = \frac{1}{f_S}$ seconds

per second we take $\frac{1}{T_S} = f_S$ samples time interval of duration T seconds $\Rightarrow$ $T f_s = \underline{\underline{TW}}$

Landau-Slepian-Pollack, 1960s

2WT-Theorem

$p = Fq$

$p = Uq$

- Q-supported in $U(=F)$
- how well can the signal be supported in the time-domain to the set P?

$$\Delta_{P,Q}(U=F) = |||D_P P_Q(F)|||_2$$

$$0 \leq \Delta_{P,Q}(U) \leq 1$$

$$\hat{q} = p = Fq$$

$I p = F q$

P frequency-domain

Q time-domain

$\Rightarrow \quad s := \underbrace{I}_{\text{identity basis}} p = \underbrace{F}_{\text{DFT basis}} q$

$\Downarrow$

replace I by A (unitary)

— " — F by B

$s = A p = B q \quad | \, A^H$

$\underbrace{A^H A}_{I} p = A^H B q$

$p = \underbrace{A^H B}_{U} q$

$$A^\# \underbrace{B B^\#}_{I} A = \underbrace{A^\# A}_{I} = I$$

$$S := \boxed{p = Uq}$$

ε_Q-concentrated to Q

ε_P- concentrated to P

Lemma 2.6. Let $U \in \mathbb{C}^{m \times m}$ and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exist a nonzero ε_P- concentrated $p \in \mathbb{C}^m$ and an ε_Q- concentrated $q \in \mathbb{C}^m$ such that $p = Uq$. Then

$$\boxed{\Delta_{P,Q}(U) \geq [1 - \varepsilon_P - \varepsilon_Q]_+ .}$$

$$\left(\Delta_{P,Q}(U) = |||D_P P_Q(U)|||_2 \right)$$

Proof.
$$\| p - P_\Omega(U) p_P \|_2 \leq \| p - P_\Omega(U) p \|_2 + \| \underbrace{P_\Omega(U) p_P - P_\Omega(U) p}_{P_\Omega(U)(p_P - p)} \|_2$$

$$\leq \| p - P_\Omega(U) p \|_2 + \underbrace{|||P_\Omega(U)|||_2}_{\leq 1} \| p_P - p \|_2$$

$$= \| Uq - U D_\Omega \underbrace{U^H U}_{I} q \|_2 + \| p_P - p \|_2$$

$$= \| q - \underbrace{D_\Omega q}_{q_\Omega} \|_2 + \| p - p_P \|_2$$

$$= \varepsilon_\Omega \|q\|_2 + \varepsilon_P \|p\|_2 .$$

$$\| P_\Omega(U) \underbrace{p_P}_{D_P p} \|_2 \geq [C \|p\|_2 - \| p - P_\Omega(U) p_P \|_2]_+$$

$$\geq [C \|p\|_2 - \varepsilon_P \|p\|_2 - \varepsilon_\Omega \|q\|_2]_+$$

$$= \|p\|_2 \,[1-\varepsilon_P-\varepsilon_Q]_+ \,. \quad \Big| \; : \|p\|_2$$

$$\Big\| P_Q(U)\, D_P \cancel{p}_{\|p\|_2} \Big\|_2 \geq [1-\varepsilon_P-\varepsilon_Q]_+$$

$$\Delta_{P,Q}(U) = \max_{x:\|x\|_2=1} \| D_P P_Q(U) x\|_2$$

$$= \max_{x:\|x\|_2=1} \| P_Q(U) D_P x\|_2$$

$$\Delta_{P,Q}(U) \geq [1-\varepsilon_P-\varepsilon_Q]_+ \,. \quad \text{q.e.d.}$$

Corollary 2.7.

$$Ap = Bq$$

$$p = \underbrace{A^H B}_{U} q$$

$$|P|\,|Q| \geq \frac{[1-\varepsilon_P-\varepsilon_Q]_+^2}{\mu^2(\langle A\,B \rangle)}$$

Proof. $[1-\varepsilon_P-\varepsilon_Q]_+ \leq \Delta_{P,Q}(u) \leq \sqrt{|P||Q|}\,\mu([I\ u])$

$$|P||Q| \geq \frac{[1-\varepsilon_p-\varepsilon_Q]_+^2}{\mu^2([I\ u])} = \frac{[1-\varepsilon_p-\varepsilon_Q]_+^2}{\underbrace{\mu^2([A\ B])}_{\max_{i,j} |\langle a_i, b_j\rangle|^2}}$$

$$\mu([I\ u]) = \mu([I\ A^H B]) = \mu([A\ B])$$

$$\begin{bmatrix} a_1^H \\ a_2^H \\ \vdots \end{bmatrix} [b_1\ b_2\ \ldots] = \begin{bmatrix} a_1^H b_1 & \ldots \\ a_2^H b_1 & \\ \vdots & \end{bmatrix}$$

$$\underbrace{|P||Q|}_{\|p\|_0 \|q\|_0} \geq \frac{1}{\mu^2([A\ B])}$$

Elad – Bruckstein which generalizes Donoho – Stark

$(A = I,\ B = F)$

$$\boxed{\|p\|_0 \, \|q\|_0 \geq m}$$

2.2.4. Noisy recovery in $(\mathbb{C}^m, \|\cdot\|_2)$

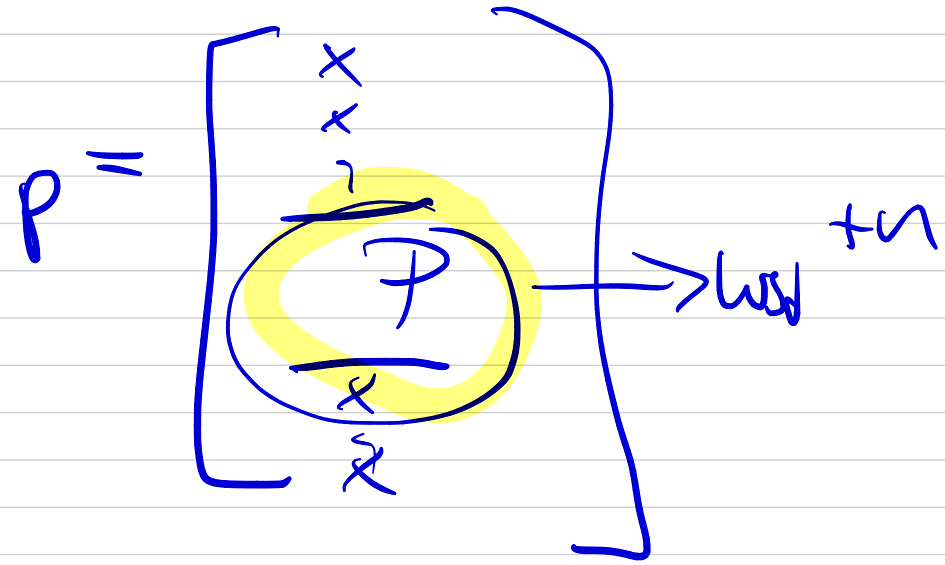
$p \in \mathbb{C}^m$

$n \in \mathbb{C}^m$

$P \subseteq \{1, \dots, m\}$

$P^c = \{1, \dots, m\} \setminus P$

$$y = p_{P^c} + n$$



Lemma 2.9. Let $U \in \mathbb{C}^{m \times m}$ be unitary, $Q \subseteq \{1, \dots, m\}$, $p \in W^{U,Q}$, and consider

$$y = p_{P^c} + n,$$

where $n \in \mathbb{C}^m$. If $\Delta_{P,Q}(U) < 1$, then there exists a matrix $L \in \mathbb{C}^{m \times m}$ such that

$$\|Ly - p\|_2 \leq C \|n_{P^c}\|_2$$

with $C = 1/(1 - \Delta_{P,Q}(U))$. In particular,

$$|P||Q| < \frac{1}{\mu^2(\{U\})}$$

is sufficient for $\Delta_{P,Q}(U) < 1$.

Proof. $L = (I - D_P P_Q(U))^{-1} D_{P^c}$

$$L p_{P^c} = (I - D_P P_Q(U))^{-1} (I - D_P) p \qquad p = P_Q(U) p \quad (p \in W^{U,Q})$$

$$= \underbrace{(I - D_P P_Q(U))^{-1}} \underbrace{(I - D_P P_Q(U)) p} = p.$$

$$|||(I - D_P P_Q(U))^{-1}|||_2 \leq \frac{1}{1 - |||D_P P_Q(U)|||_2}$$

$$= \frac{1}{1 - \Delta_{P,Q}(L)}$$

$$\left(\begin{aligned} &(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \quad \text{Neumann series} \\ &|||(I-A)^{-1}|||_2 \leq \underbrace{\sum_{k=0}^{\infty} |||A|||^k} = \frac{1}{1 - |||A|||} \end{aligned} \right)$$

$$\|Ly - \rho\|_2 = \|\underbrace{L\rho_{pc}} + Ln - \underbrace{\rho}\|_2 = \|Ln\|_2$$

$$y = \rho_{pc} + n$$

$$= \|Ln_{pc}\|_2 \leq |||L|||_2 \, \|n_{pc}\|_2$$

$$\leq \underbrace{\frac{1}{1 - \Delta_{P,Q}(L)}}_{=C} \|n_{pc}\|_2 . \qquad \text{q.e.d.}$$

$P = \{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \}$, $U = F$

$Q = \{ l+1, \ldots, l+n \} \leftarrow$

$\Delta_{P,Q}(F) = \sqrt{n/m}$, n divides m $\Rightarrow$ stable recovery of p is possible as soon as $n \leq m/2$.

$m \geq 2n$

ambient space has dimension m

p is n-sparse in F

we are missing n entries in the noisy observation y

$m-n$ entries are seen

$\underbrace{m-n}_{\#\text{measurements}} \geq 2n - n = n$