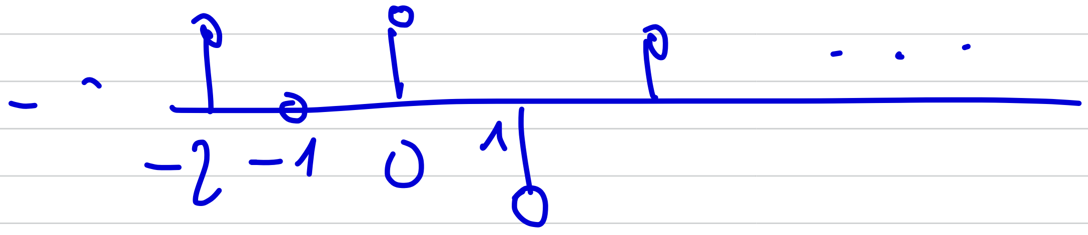
Compressed Sensing

Discrete Fourier Transform

$x[n], \quad n \in \mathbb{Z}$

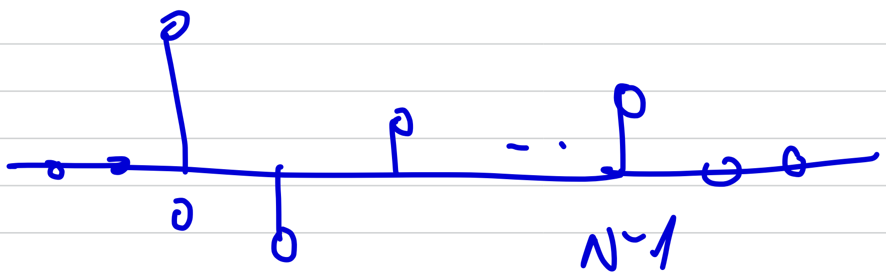
$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-i 2\pi \theta n}, \quad 0 \le \theta < 1$$



$$\hat{x}(\theta+1) = \sum_{n=-\infty}^{\infty} x[n] \underbrace{e^{-i 2\pi (\theta+1) n}}_{e^{-i 2\pi \theta n}}$$

$$= \hat{x}(\theta), \quad \theta \in [0,1)$$

$x[n] = 0, \quad n < 0, \; n \ge N$



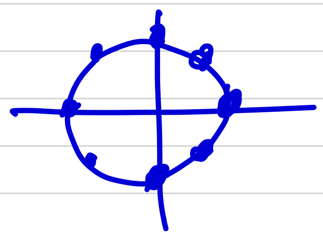
$$\hat{x}(\theta) = \sum_{n=0}^{N-1} x[n] e^{-i 2\pi \theta n}, \quad \theta \in [0,1)$$

DTFT ... $$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i2\pi\theta n}$$

IDTFT ... $$x[n] = \int_0^1 \hat{x}(\theta)\, e^{i2\pi\theta n}\, d\theta$$

$$\int_0^1 \hat{x}(\theta)\, e^{i2\pi\theta n}\, d\theta = \int_0^1 \sum_{n'=-\infty}^{\infty} x[n']\, e^{-i2\pi\theta n'}\, e^{i2\pi\theta n}\, d\theta$$

$$= \sum_{n'=-\infty}^{\infty} x[n'] \underbrace{\int_0^1 e^{i2\pi\theta(n-n')}\, d\theta}_{= \begin{cases} 1, & n = n' \\ 0, & n \neq n' \end{cases}}$$



$$= \sum_{n'=-\infty}^{\infty} x[n']\, \delta[n-n'] = x[n]$$

$$x[n] = 0, \quad n < 0, \; n \geq N$$

$$\hat{x}(\theta) = \sum_{n=0}^{N-1} x[n] \, e^{-i 2\pi \theta n}$$

$$\underbrace{\hat{x}\left(\frac{k}{N}\right)}_{\hat{x}[k]} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \, e^{-i 2\pi \frac{k}{N} n}, \quad k = 0, 1, \ldots, N-1$$

$$\underbrace{\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \vdots \\ \hat{x}[N-1] \end{bmatrix}}_{\hat{x}} = \frac{1}{\sqrt{N}} \underbrace{\begin{array}{c|c} & n \rightarrow \\ \hline k \downarrow & \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-i2\pi/N} & & \\ \vdots & & \frac{1}{\sqrt{N}} \left[e^{-i 2\pi \frac{kn}{N}} \right]_{k,n} & \\ 1 & & & \end{bmatrix} \end{array}}_{\substack{\text{DFT-matrix} \\ F_N}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{x}, \quad \hat{x} = F_N x$$

$$\frac{1}{\sqrt{N}} \, \omega_N^{kn}$$

$$\omega_N = e^{-i 2\pi / N}$$

$$\frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{-i2\pi (r/N)\cdot l} \; e^{i2\pi (s/N)\cdot l}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} e^{i2\pi \frac{s-r}{N} l} = \frac{1}{N} \frac{1 - \overbrace{e^{i2\pi \frac{s-r}{N} \cdot N}}^{1}}{1 - e^{i2\pi \frac{s-r}{N}}} = 0, \; r \neq s$$

$$= \frac{1}{N} \cdot N = 1, \quad r = s$$

F_N ... unitary matrix

$$F_N F_N^H = F_N^H F_N = I_N$$

$$\hat{x} = F_N x \quad \Big| F_N^H \cdot$$

$$F_N^H \hat{x} = \underbrace{F_N^H F_N}_{I} x \Rightarrow x = F_N^H \hat{x}$$

$\hat{x} = F_N x$ ... DFT

$x = F_N^H \hat{x}$ ... IDFT

$$\hat{x}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi kn/N} \quad , \quad k = 0,1,\ldots,N-1$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}[k]\, e^{i2\pi kn/N} \quad , \quad n = 0,1,\ldots,N-1$$

$\hat{x}[k] = \hat{x}[k+N]$

$x[n] = x[n+N]$

so far: critical sampling

Oversampling:

$$\hat{x}(\theta) = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi\theta n} \quad , \quad \theta \in [0,1)$$

$$\hat{x}\left(\frac{k}{M}\right) = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi \frac{k}{M} n} \quad , \quad k = 0,1,\ldots,M-1$$

$M > N$

$\hat{x}[0]$ $\hat{x}[1]$

$$\begin{bmatrix} \hat{x}(0) \\ \hat{x}(1/M) \\ \vdots \\ \hat{x}\left(\frac{M-1}{M}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \left[e^{-i2\pi \frac{kn}{M}} \right]_{\substack{k=0,1,\dots,M-1 \\ n=0,1,\dots,N-1}} \end{bmatrix}}_{M \times N} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$\hat{x}[M-1]$

$$\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \vdots \\ \hat{x}[M-1] \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & - & \cdots & 1 \\ 1 & e^{-i2\pi/M} & \cdots & \\ \vdots & & \vdots & \\ 1 & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$\underline{\underline{F_0}}$ first N columns of F_M

T

$$\underbrace{\overset{N\times M}{T^H}\;\overset{M\times N}{T}}_{S} = I_N$$

$$\underset{M\times 1}{\hat{x}} = \underset{M\times N}{F_0}\;\underset{N\times 1}{x}$$

$$F_0^H F_0 = I$$

$$x = \underbrace{(F_0^H F_0)^{-1} F_0^H}_{F_0^\#} \hat{x}$$

$$= \underbrace{(F_0^H F_0)^{-1} F_0^H F_0}_{I} x = x \checkmark$$

$$x = F_0^{\#} \hat{x}$$

$$F_0^{\#} = \underbrace{(F_0^H F_0)}_{I}^{-1} F_0^H = F_0^H$$

$$x = F_0^H \hat{x}$$

$$x[n] = \frac{1}{\sqrt{M}} \sum_{k=0}^{N-1} \hat{x}[k]\, \omega_M^{-kn}$$

$$\hat{x}[k] = \frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} x[n]\, \omega_M^{kn}$$

Undersampling

$$\hat{x}\left(\frac{k}{M}\right) = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi kn/M} \quad , \quad k = 0, 1, \ldots, M-1$$

$$M < N$$

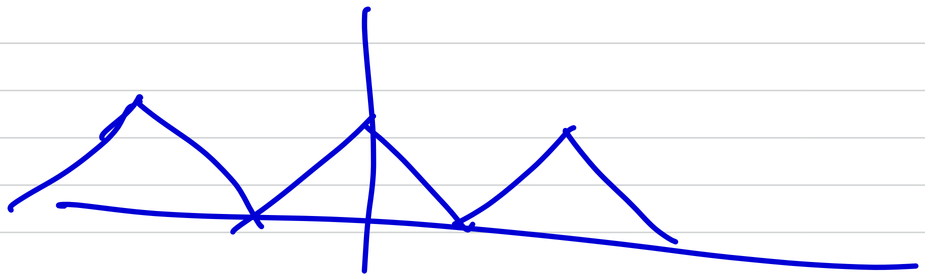
$$\hat{x} = \underbrace{F_u}_{m \times N} x$$

$x + n$

$\in \ker(F_u)$

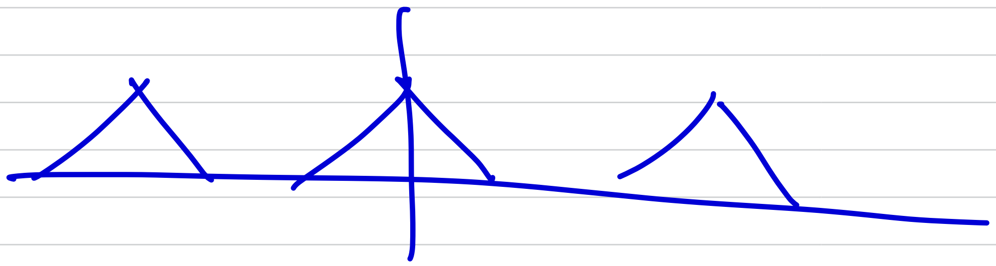
$$F_u(x+n) = F_u x + \underbrace{F_u n}_{=0}$$

there are infinitely many solutions



critical sampling

$$\hat{x} = \underbrace{F_N}_{N \times N} x$$

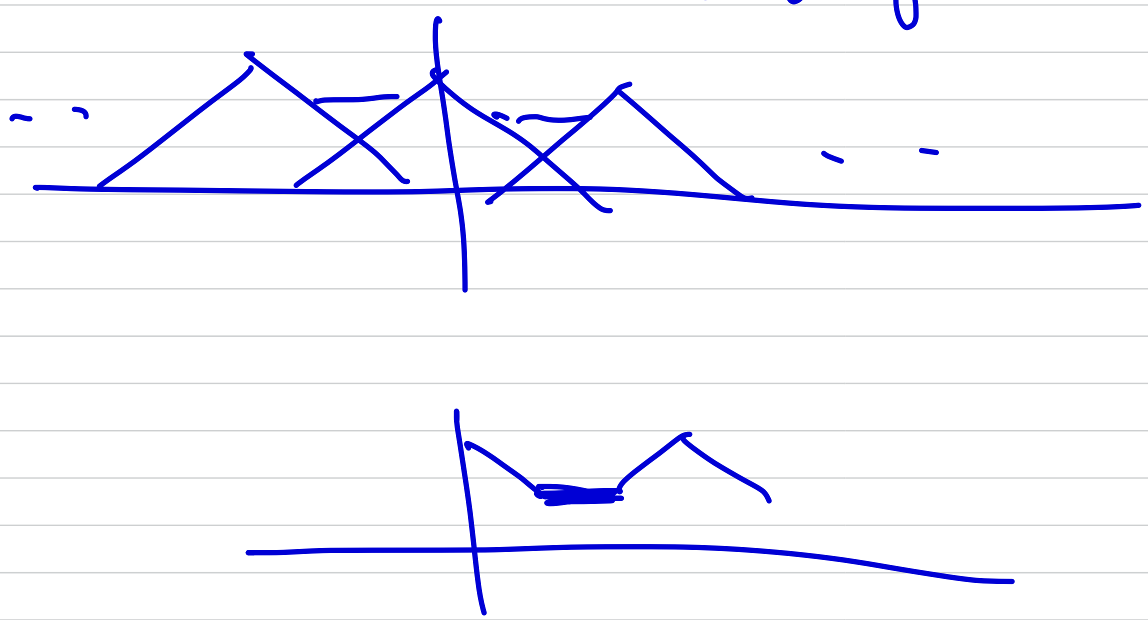


oversampling

$$\hat{x} = F_M x$$

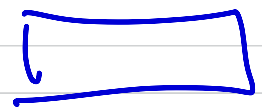
$M \times N,\ M > N$

aliasing



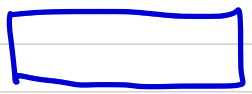
undersampling $\quad \hat{x} = F_u x$

$M \times N, \; M < N$



3.2 Compressed Sensing

$$y = D x$$



applications: whenever we can not sample at a rate that is high enough to recover the signal of interest

e.g.

- MRI, CT

- acquisition difficult or costly
  geophysics

- earth observation

- A/D conversion
  noise shaping, sigma-delta converters

The structure we are going to be interested in is sparsity, i.e., we consider vectors x that have only a small no. of their entries nonzero

spectrum

$$\underset{m\times 1}{y} = \underbrace{F_n^H}_{D}\; x$$

sparse

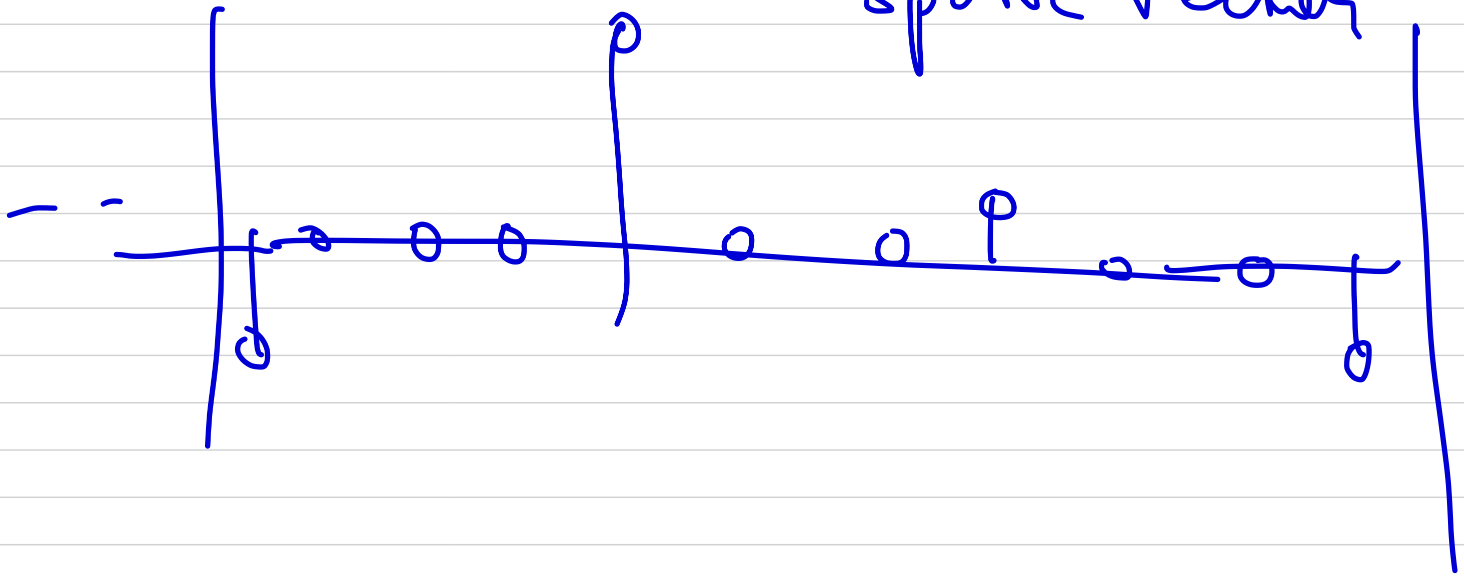
x ... $n\times 1$

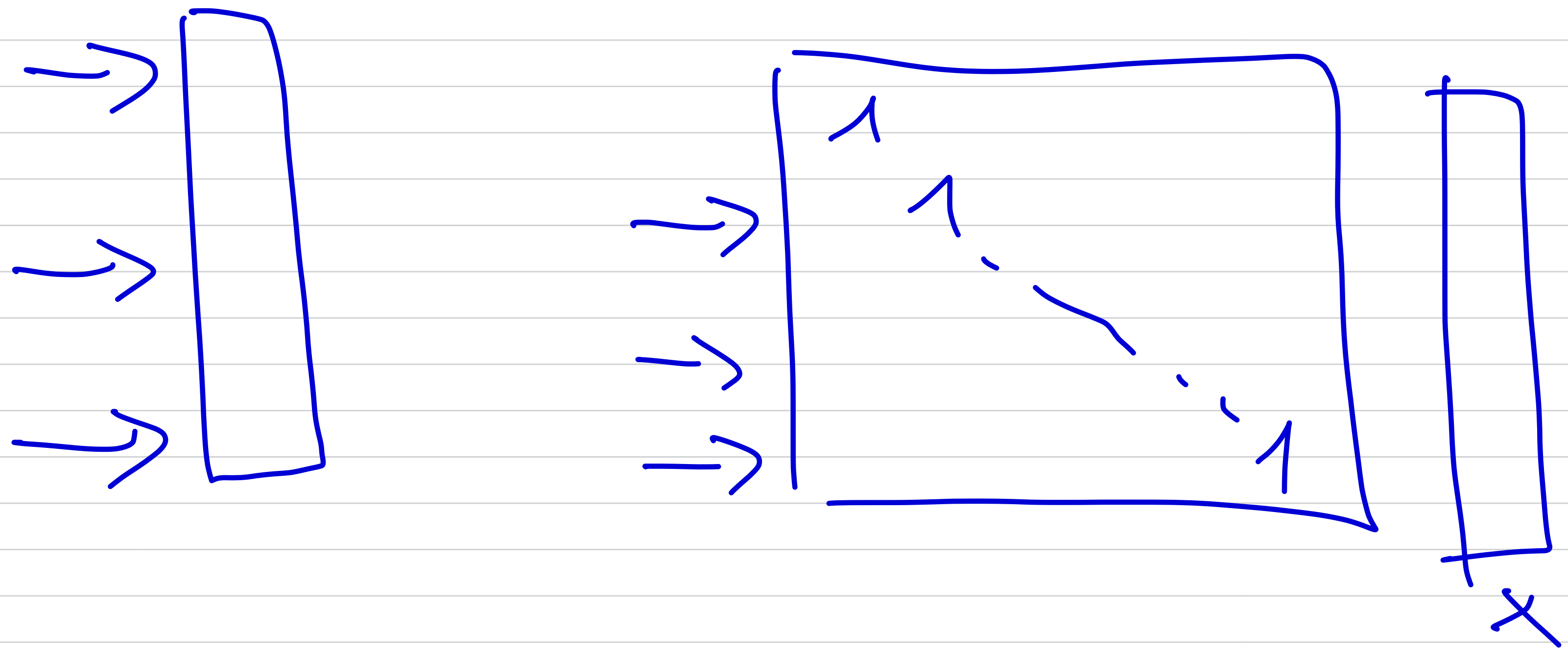
1. assume knowledge of support set

$\|x\|_0 = s$

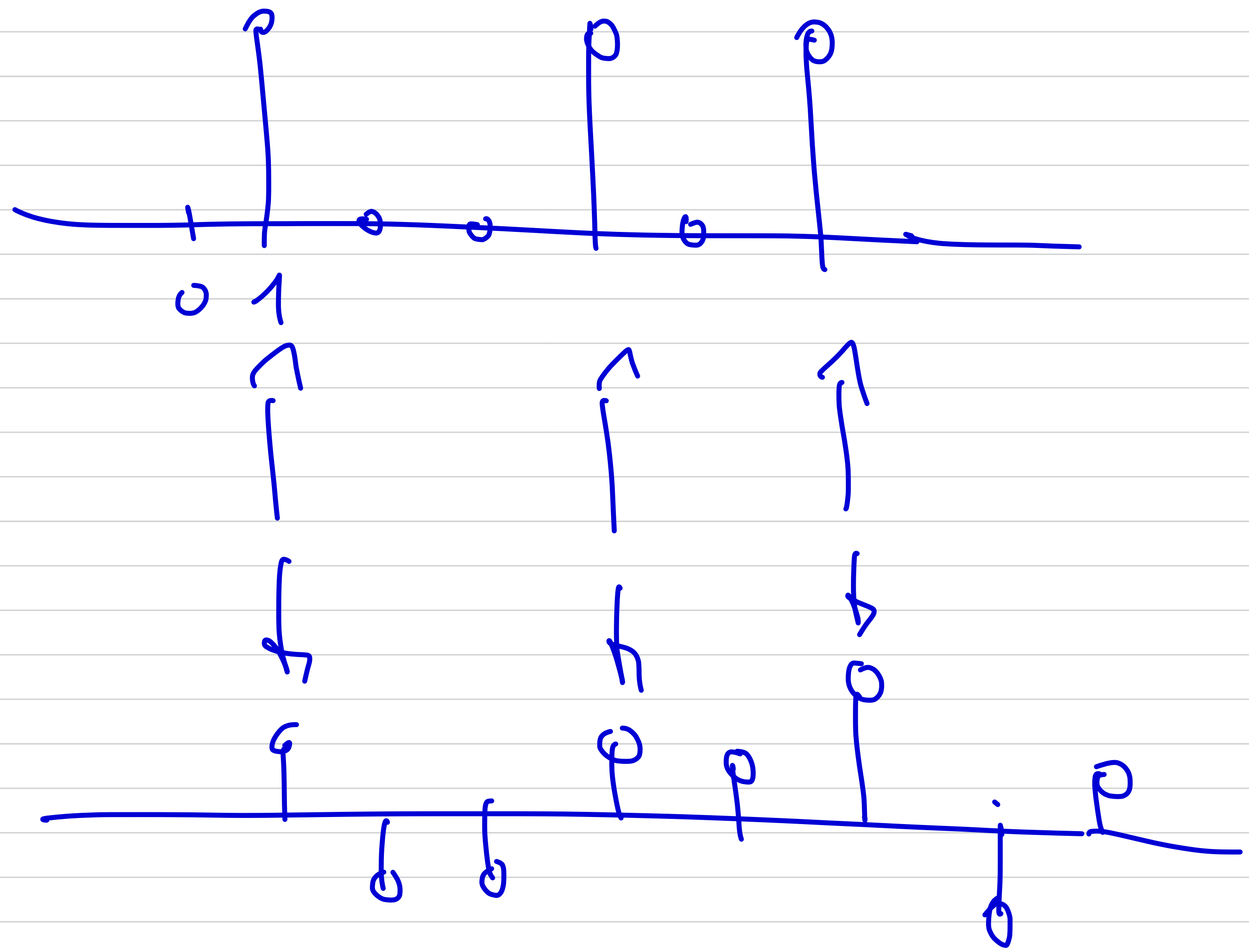
no. of nonzero coefficients

sparse vector





$$\begin{bmatrix} 0 & 1 & 0 & - & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x(0) \\ \\ \\ \end{bmatrix}}_{X} = \begin{bmatrix} x(1) \\ x(4) \\ x(6) \end{bmatrix}$$



1. we know the support set of x

$$y = \underbrace{F_n^{\#}}_{m = s \text{ samples}}\, x = \underbrace{A}_{\substack{m \times m \\ s \times s}}\, x$$

pick first s rows of F_n^{H} $\Rightarrow$ guarantees that every resulting $s \times s$ submatrix is of full rank

$\left(e^{i 2\pi \frac{s_1}{n}}\right)^0 = z_1$

z_2

$$s \text{ rows} \left\{ \begin{bmatrix} 1 & 1 & \cdots \\ e^{+i 2\pi \frac{s_1}{n}} & e^{+i 2\pi \frac{s_2}{n}} & \cdots \\ e^{+i 2\pi \frac{2 s_1}{n}} & e^{+i 2\pi \frac{2 s_2}{n}} & \\ \vdots & \vdots & \end{bmatrix} \right. = \overbrace{\begin{bmatrix} 1 & 1 & \cdots \\ z_1^1 & z_2^1 & \cdots \\ z_1^2 & z_2^2 & \cdots \\ \vdots & \vdots & \\ z_1^{s-1} & z_2^{s-1} & \cdots \end{bmatrix}}^{\text{Vandermonde}}$$

row, column

$$e^{-i 2\pi \frac{rs}{n}}$$

$s_1, s_2, \ldots$

$z_1 = e^{i 2\pi \frac{s_1}{n}}$

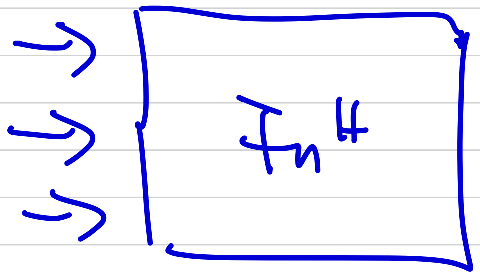
$z_2 = e^{i 2\pi \frac{s_2}{n}}$

$\vdots$

Vandermonde matrix has full rank whenever the generators z_i are pairwise distinct, i.e., they are all different

2. we don't know the support set of x, but we do know $\|x\|_0 \le s$
want to make sure that no two different x-vectors lead to the same observation vector

$$y = \overbrace{D}^{\boxed{}} x$$



$$y = D x_1$$
$$y = D x_2$$

, $x_1 \neq x_2$ lead to same y

$$0 = y - y = Dx_1 - Dx_2 = D(\underbrace{x_1 - x_2}_{(2s)\text{-sparse}})$$