wavelet coefficients

$$\underbrace{W \cdot \text{image}}$$

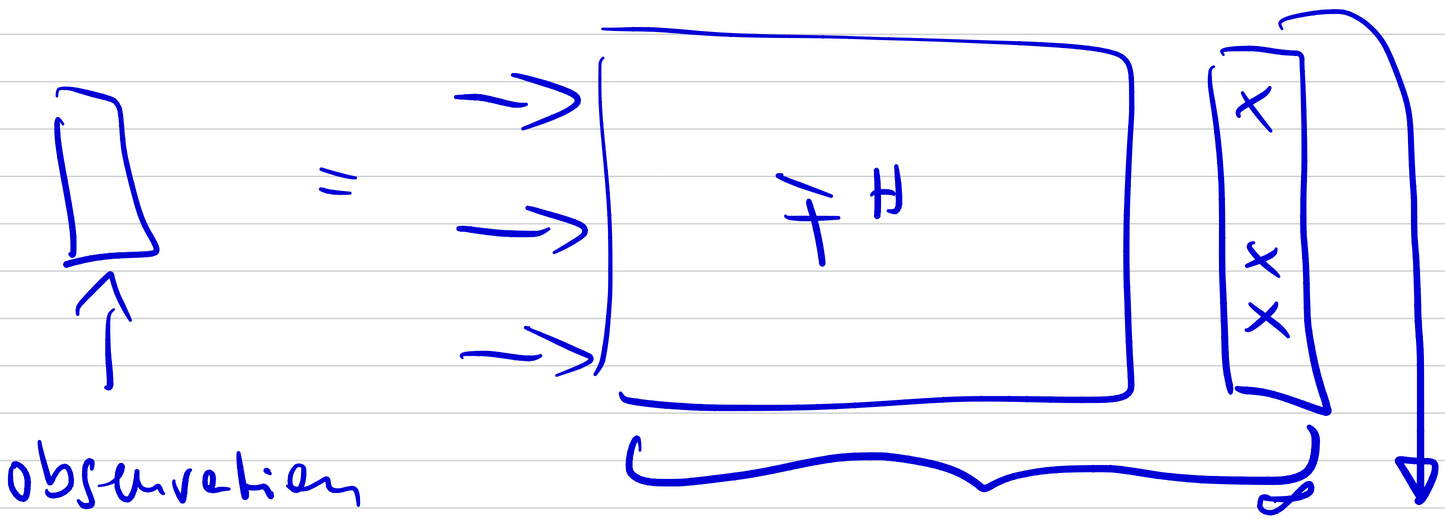
$$y = \overbrace{W^H}^{A} x \qquad \leftarrow \text{sparse}$$

$10^6 \times 10^6$ orth. wavelet transform

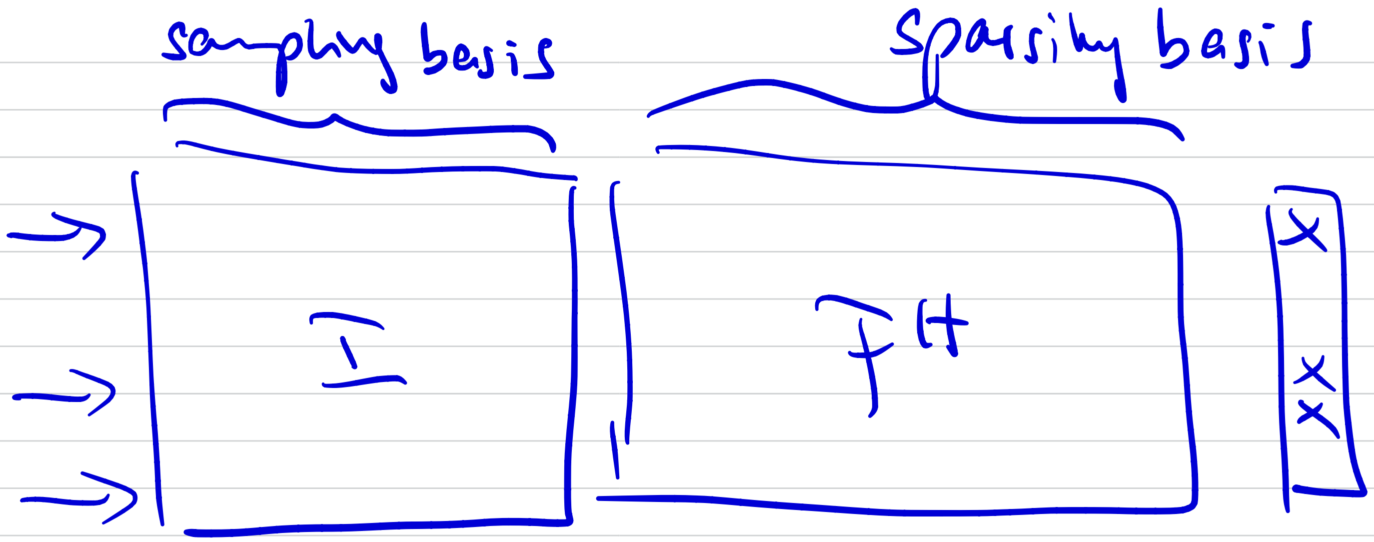
$$y = D x$$

$$D = [d_1 \ d_2 \ \ldots \ d_n]$$

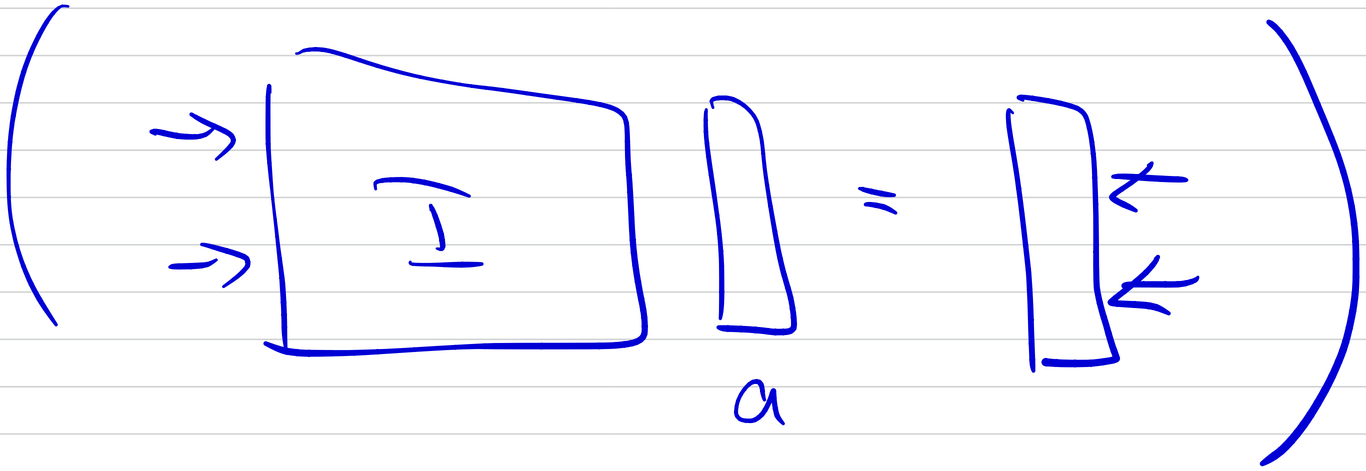
$$y = Dx = \sum_{i=1}^{s} d_i x_i$$

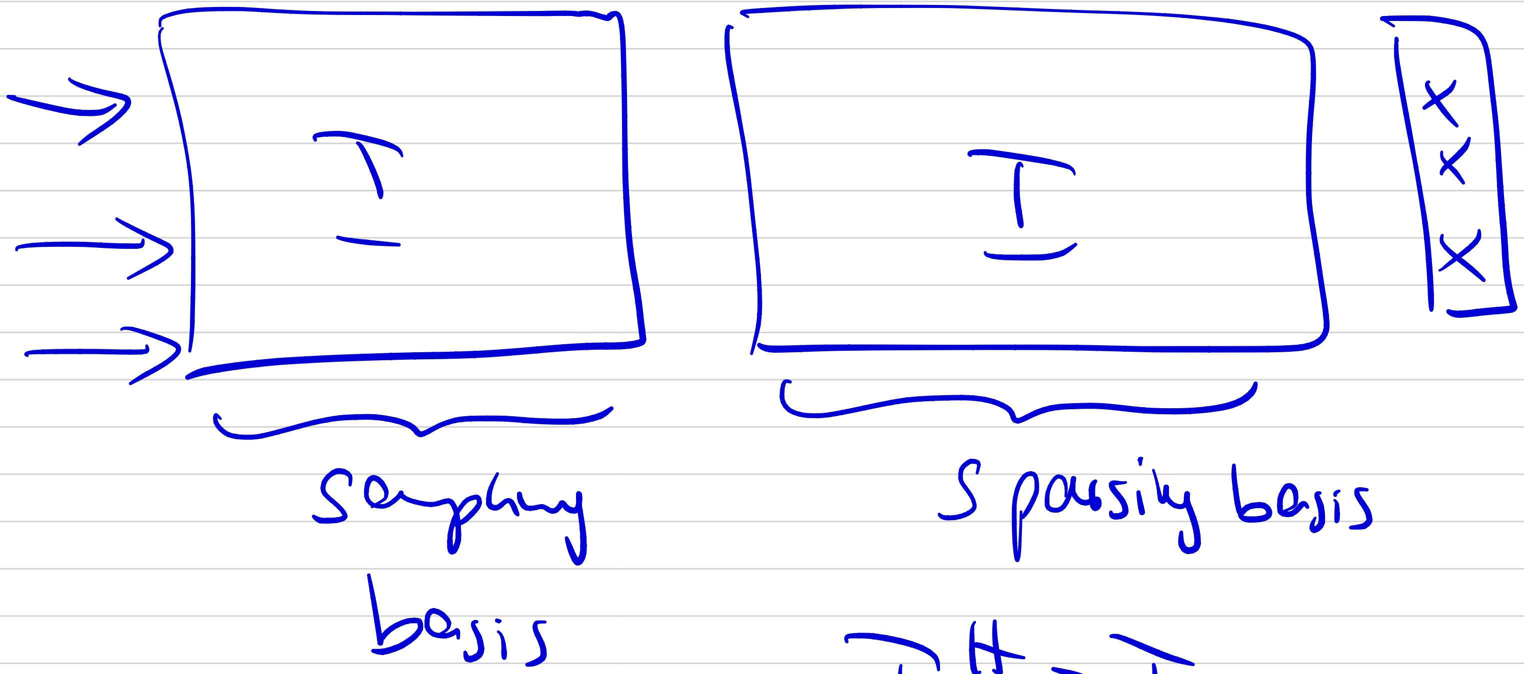


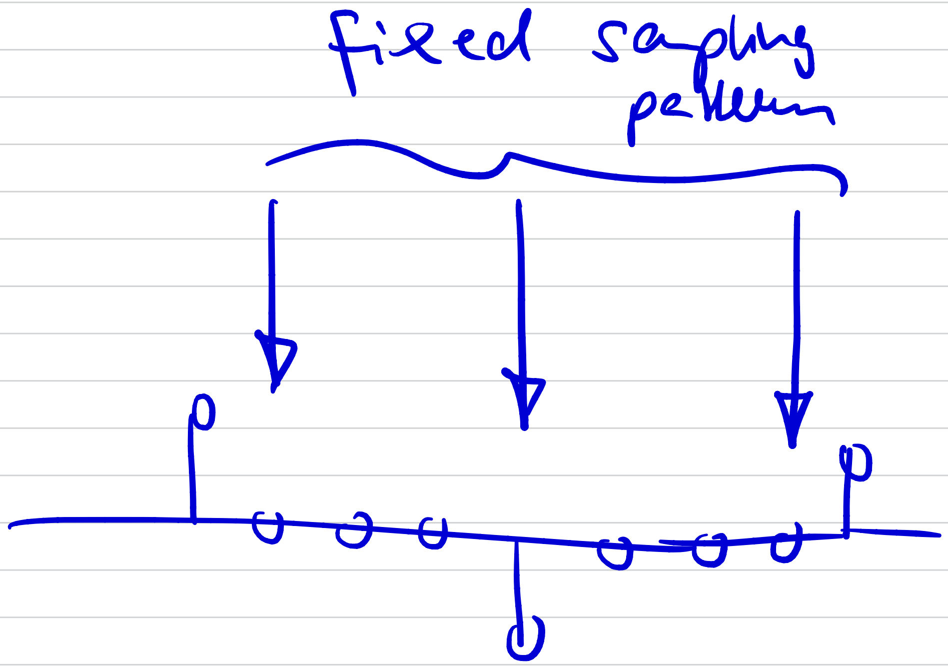
$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{i 2\pi f t} \, df$$


sampling basis
sparsity basis
I
F^H

$y = Dx$

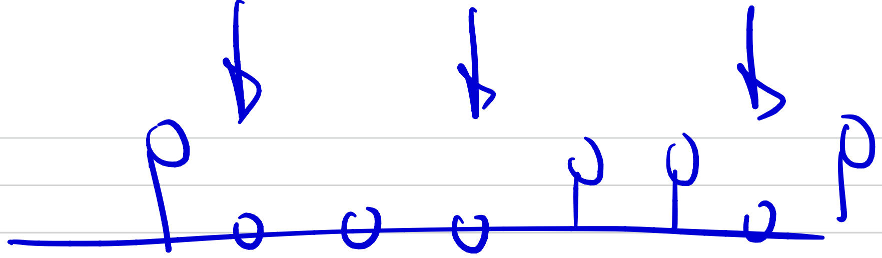

I
=
a

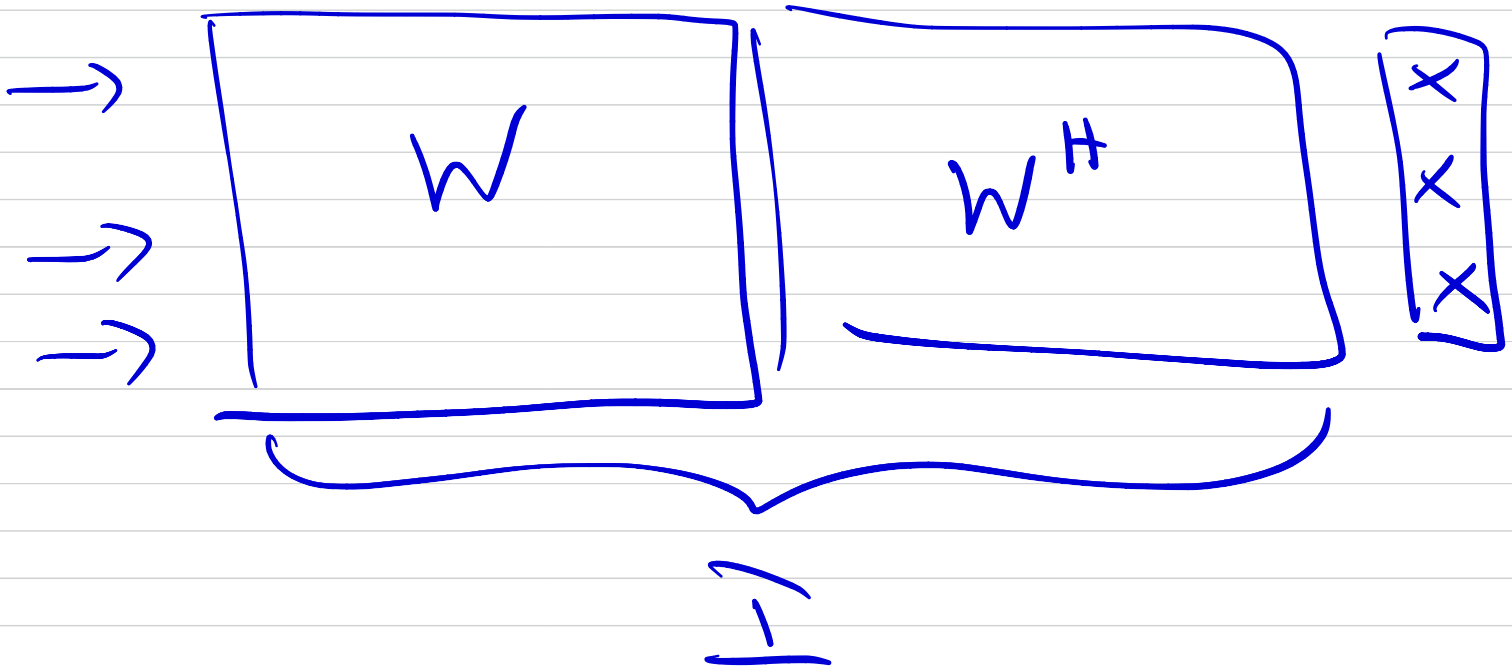

I
I
sampling basis
sparsity basis

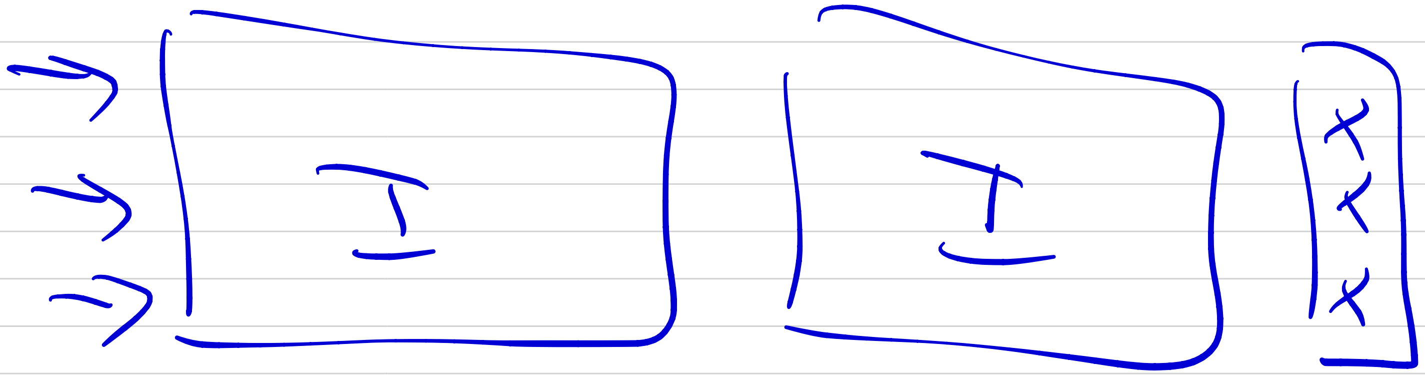

fixed sampling pattern

$I^H = I$

$$W^H = W$$







3.2.2. The general problem

$$y = Dx$$

Uniqueness of recovery

$$y_1 = Dx_1 \qquad \|x_1\|_0 \le s, \; \|x_2\|_0 \le s$$
$$y_2 = Dx_2$$

if $x_1 \neq x_2 \Rightarrow y_1 \neq y_2$

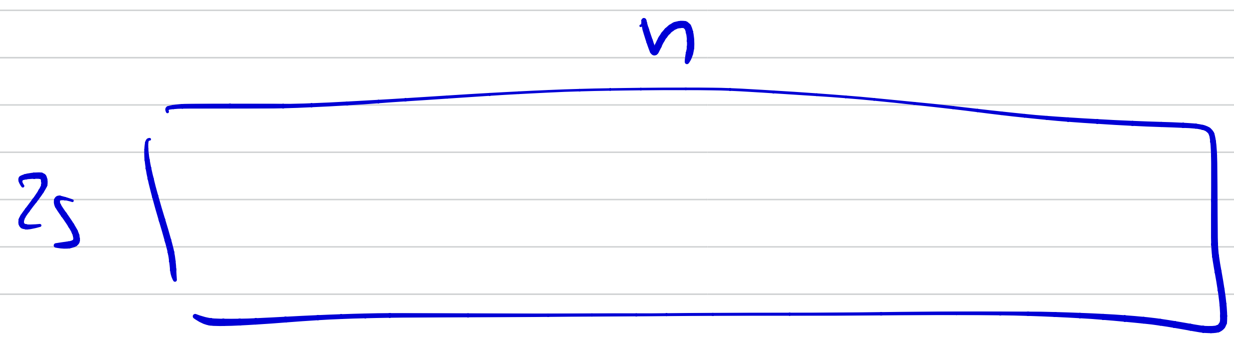
want to ensure that the following does not happen

$$y_1 = y_2 = y : \; 0 = y - y = Dx_1 - Dx_2 = D\underbrace{(x_1 - x_2)}_{\|x_1 - x_2\|_0 \le 2s}$$

- need to ensure that every set of $2s$ or fewer columns of D is linearly independent.

Def. 3.1. The spark of a matrix A, denoted by spark(A), is defined as the cardinality of the smallest set of linearly dependent columns of A.

To get uniqueness of recovery, we need D to have $\geq 2s$ rows, this is often referred to as "sampling" at the Landau rate.



Uniqueness of recovery $\Rightarrow$ $s < \frac{spark(D)}{2}$

3.3. The (P0) recovery algorithm

$y = Dx$ ← true x

If x is s-sparse and

$$s < \frac{spark(D)}{2}$$

We can recover x through a combinatorial search:

(P0) find $\arg\min \|\tilde{x}\|_0$ subject to $y = D\tilde{x}$

Proof of (P0) delivering the correct solution:

Suppose that $\|x\|_0 \leq s$ and $s < \frac{\text{spark}(D)}{2}$. Consider $\tilde{x} \neq x$ with $\|\tilde{x}\|_0 \leq s$ and $y = D\tilde{x}$.

$$0 = y - y = Dx - D\tilde{x} = D(\overbrace{\underbrace{x - \tilde{x}}_{2s\text{-sparse}}}^{\neq 0}) \Rightarrow \text{contradiction.}$$

$$\mu(D) = \max_{r \neq l} |\langle d_l, d_r \rangle|$$

Theorem 3.2. (P0) applied to $y = Dx$ recovers x if ($D \in \mathbb{C}^{m \times n}$)

$$\underbrace{\|x\|_0 \leq s}_{\text{sparsity is } s} < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right)$$

Proof. show that $\text{spark}(D) \geq 1 + 1/\mu(D)$

$$\left(s < \frac{\mathrm{spark}(D)}{2} \right)$$

consider $x \in \mathbb{C}^n$, with $\|x\|_0 = \mathrm{spark}(D)$ and $Dx = 0$

$$Dx = \sum_{r=1}^{n} d_r x_r = 0$$

$$d_\ell x_\ell = - \sum_{\substack{r=1 \\ r \neq \ell}}^{n} d_r x_r \quad \Big| \; d_\ell^H \cdot$$

$$\underbrace{d_\ell^H d_\ell}_{1} x_\ell = - \sum_{\substack{r=1 \\ r \neq \ell}}^{n} d_\ell^H d_r x_r$$

$$|x_\ell| = \Big| \sum_{\substack{r=1 \\ r \neq \ell}}^{n} d_\ell^H d_r x_r \Big| \leq \sum_{\substack{r=1 \\ r \neq \ell}}^{n} |d_\ell^H d_r| \, |x_r| \leq$$

$$\leq \mu(D) \sum_{\substack{r=1 \\ r \neq \ell}}^{n} |x_r|$$

add $\mu(D)|x_\ell|$

$$(1+\mu(D))|x_\ell| \leq \mu(D) \underbrace{\sum_{r=1}^{n} |x_r|}_{\|x\|_1}$$

sum over all ℓ for which $x_\ell \neq 0$

$$(1+\mu(D))\cancel{\|x\|_1} \leq \mu(D)\cancel{\|x\|_1}\,\mathrm{spark}(D)$$

$$\mathrm{spark}(D) \geq 1 + \frac{1}{\mu(D)}. \qquad \square$$

3.4. Basis pursuit (BP)

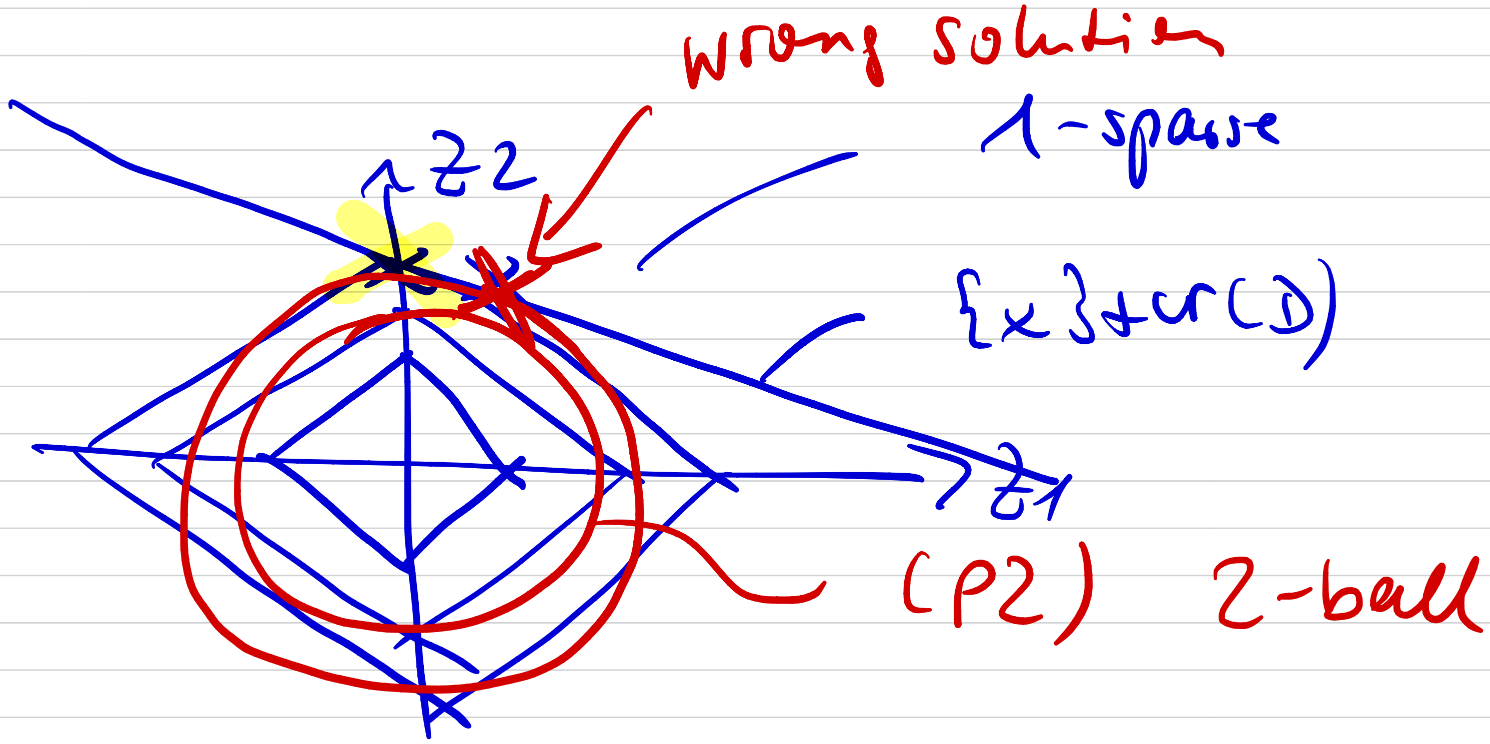
(P1) Find arg min $\|x\|_1$ subject to $y = Dx$

convex optimization problem $\Rightarrow$ numerous algorithms available why does ℓ_1-reconstruction work?

$$\arg\min \|\hat{x}\|_1 \quad \text{subject to} \quad y = D\hat{x}$$

$$\Updownarrow$$

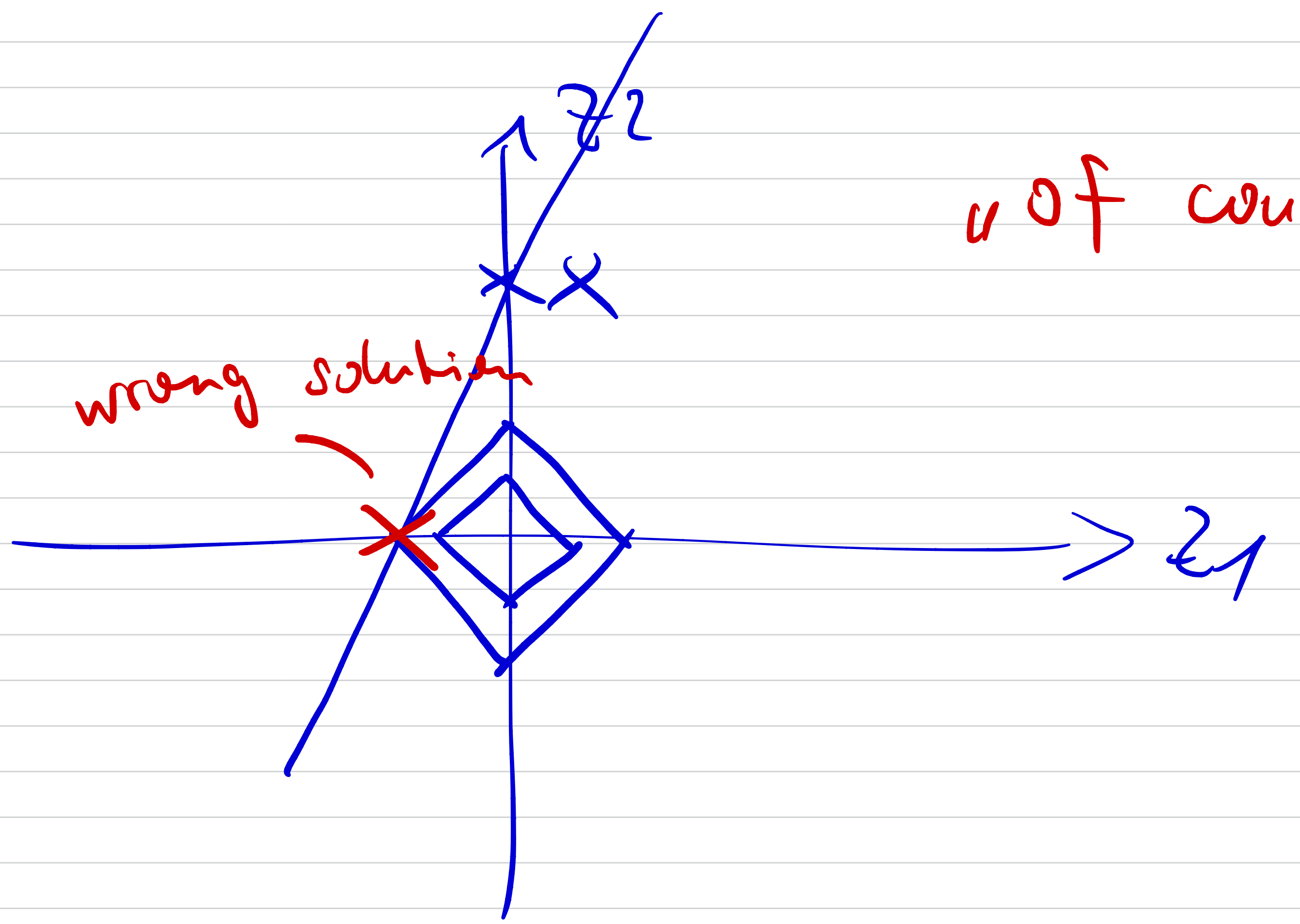
$$\arg\min \|\hat{x}\|_1 \quad \text{subject to} \quad \hat{x} \in (\{x\} + \ker(D))$$



$$\|z\|_1 = |z_1| + |z_2| \le c$$

$$|z_2| \le c - |z_1|$$

The 1-norm does not always sparsify



"of course"

Def. 3.3. We denote

$$P_1(S,D) := \max_{x \in \mathcal{N}(D),\, x \neq 0} \overbrace{\frac{\sum_{s \in S} |x_s|}{\sum_s |x_s|}}^{\text{fraction of } \ell_1\text{-norm on } S}$$

Th. 3.4. Arbitrarily fix x with support set S and let $y = Dx$. If $P_1(S,D) < 1/2$, then x is the unique solution to

(P1): find any min $\|x\|_1$ subject to $y = Dx$

Proof. need to establish that for all $\alpha \in \mathcal{N}(D)$)

$$\underbrace{\sum_g |x_g + \alpha_g|}_{\text{1-norm of competing solutions}} > \underbrace{\sum_g |x_g|}_{\text{1-norm of correct solution}}$$

$|a+b| \geq |a| - |b|$, reverse Δ-inequ.

$$\sum_g |x_g + \alpha_g| = \sum_{g \notin S} |x_g + \alpha_g| + \sum_{g \in S} |x_g + \alpha_g|$$

$$= \sum_{g \notin S} |\alpha_g| + \underbrace{\sum_{g \in S} |x_g + \alpha_g|}_{\geq \sum_{g \in S} |x_g| - \sum_{g \in S} |\alpha_g|}$$

$$\geq \sum_{s\notin S} |\alpha_s| + \cancel{\sum_{s\in S} |x_s|} - \sum_{s\in S} |\alpha_s|$$

$$> \cancel{\sum_{s\in S} |x_s|}$$

$$\Rightarrow \sum_{s\notin S} |\alpha_s| > \sum_{s\in S} |\alpha_s| \quad \Big| + \sum_{s\in S} |\alpha_s|$$

$$\sum_{s} |\alpha_s| > 2 \sum_{s\in S} |\alpha_s|$$

$$\frac{\sum_{s\in S} |\alpha_s|}{\underbrace{\sum_{s} |\alpha_s|}_{}} < \frac{1}{2}$$