$$\sqrt{m} \gtrsim s$$

$$\boxed{m \gtrsim s^2}$$

square-root bottleneck

1. $m \geq s^2$
2. $m \sim s \log(n)$ $\Leftarrow$ JL-Lemma
3. $\boxed{m \geq 2s}$

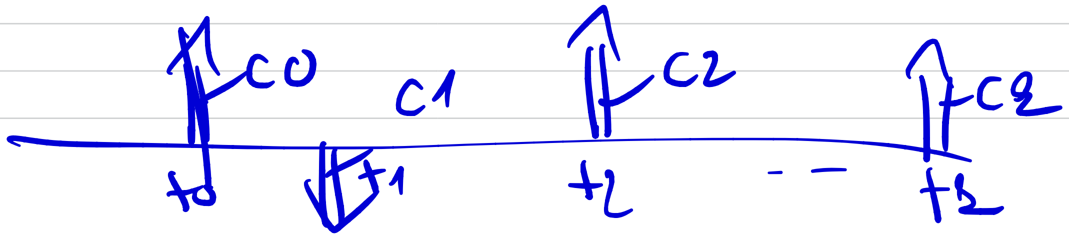
Chapter 4: Sampling of signals with finite rate of innovation

$$x(t) = \sum_{k=0}^{K-1} c_k \, \delta(t - t_k), \quad 0 \leq t_k \leq \tau$$

$\uparrow$ Dirac delta function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{else} \end{cases}$$

$$\int \delta = 1$$



$$\hat{x}(f) = \sum_{k=0}^{K-1} c_k e^{-i2\pi t_k f} \cdot 1 \quad , \quad \text{not band-limited}$$

$\Rightarrow$ need to sample the signal at a rate of $f_s = \infty$ to recover it.

"sampling theorem tells us that this signal can not be recovered from its samples". really?

$x(t)$ is not band-limited, but it is highly structured and it is fully (modulo K) determined by the t_k and the c_k, i.e., it is fully determined by $2K$ parameters.

Q: Can we take measurements of $x(t)$ in the frequency domain such that t_k, c_k (and hence $x(t)$) can be recovered? If yes, how many measurements do we need to take?

$$d_n = \frac{1}{\tau} \int_0^{\tau} \sum_{k=0}^{K-1} c_k \delta(t-t_k) e^{-i2\pi \frac{n}{\tau} t} dt = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \int_0^{\tau} \delta(t-t_k) e^{-i2\pi \frac{n}{\tau} t} dt$$

$\uparrow$

Fourier series coefficients

(sifting property: $\int \delta(t-\tau) f(t)\, dt = f(\tau)$)

$$= \frac{1}{\tau} \sum_{\ell=0}^{K-1} c_\ell \, e^{-i 2\pi \frac{n}{\tau} t_\ell}$$

$$\boxed{d_n = \frac{1}{\tau} \sum_{\ell=0}^{K-1} c_\ell \, e^{-i 2\pi \frac{n}{\tau} t_\ell}}$$

$$x(t) = \sum_{n \in \mathbb{Z}} d_n \, e^{i 2\pi \frac{n}{\tau} t} =$$

$$= \sum_{n \in \mathbb{Z}} \left(\frac{1}{\tau} \sum_{\ell=0}^{K-1} c_\ell \, e^{-i 2\pi \frac{n}{\tau} t_\ell} \right) e^{i 2\pi \frac{n}{\tau} t}$$

Do we need infinitely many Fourier series coefficients to recover the t_ℓ & the c_ℓ, $\ell = 0, 1, \ldots, K-1$?

Sparsity level is $2K = s$
no. of measurements m = no. of Fourier series coefficients
ambient dimension: N/A

Given

$$d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k e^{-i 2\pi n \frac{t_k}{\tau}} \quad , \quad n = ?$$

recover c_k, t_k, $k = 0, 1, \ldots, K-1$.

<u>Berlekamp–Massey algorithm</u> $\longleftarrow$

<u>Peterson–Gorenstein–Zierler alg.</u>

"annihilating filter method"

$$A(z) = \sum_{m=0}^{K} a_m z^{-m} \quad , z \in \mathbb{C}$$

Interlude: Filter theory

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = (x * h)[n] = \sum_{\ell=-\infty}^{\infty} x[\ell] h[n-\ell]$$

LTI ... linear time-invariant

$$= \sum_{\ell=-\infty}^{\infty} h[\ell] x[n-\ell]$$

$$Y(z) = X(z)\,H(z)$$

$$Y(z) = \sum_n y[n]\,z^{-n} \quad \leftarrow \text{ Z-transform}, \qquad \begin{array}{l} y[n] \circ\!\!-\!\!\bullet\ Y(z) \\ y[n-1] \circ\!\!-\!\!\bullet\ z^{-1}Y(z) \end{array}$$

$$A(z) = \sum_{m=0}^{K} a_m z^{-m} = \prod_{k=0}^{K-1}\left(1 - e^{-i2\pi\frac{t_k}{\tau}} z^{-1}\right), \quad \text{zeros} = e^{i2\pi\frac{t_k}{\tau}}$$

step specifies the a_m

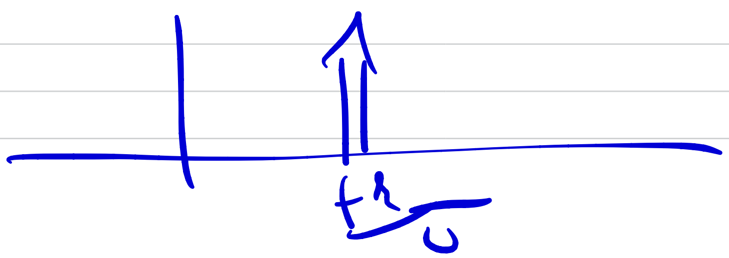
$$\delta\left(\theta - \frac{t_k}{\tau}\right)$$

$$\delta(\theta) \quad \left(1 - e^{-i2\pi\frac{t_k}{\tau}} z^{-1}\right) \circ\!\!-\!\!\!-\!\!\bullet\ \delta[n] - e^{-i2\pi\frac{t_k}{\tau}}\,\delta[n-1]$$

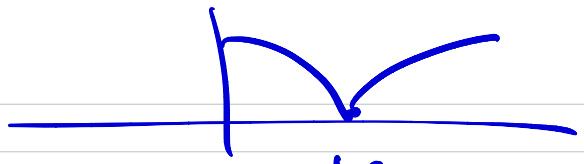
$$1 \cdot e^{-i2\pi\frac{t_k}{\tau}n} * \left(\delta[n] - e^{-i2\pi\frac{t_k}{\tau}}\,\delta[n-1]\right)$$

$$= e^{-i2\pi\frac{t_k}{\tau}n} - e^{-i2\pi\frac{t_k}{\tau}}\, e^{-i2\pi\frac{t_k}{\tau}(n-1)}$$

$$= e^{-i2\pi\frac{t_k}{\tau}n} - \underbrace{e^{-i2\pi\frac{t_k}{\tau}}\, e^{i2\pi\frac{t_k}{\tau}}}_{1}\, e^{-i2\pi\frac{t_k}{\tau}n} = \underline{\underline{0}}$$



$$\delta[n] - e^{-i2\pi\frac{t_k}{\tau}}\,\delta[n-1] \circ\!\!-\!\!\bullet$$



$$d_\ell * a_\ell = 0 \qquad\qquad a_\ell \circ\!\!-\!\!\bullet A(z)$$

$$d_\ell = \frac{1}{\tau}\sum_{k=0}^{K-1} c_k e^{-i2\pi\frac{t_k}{\tau}\ell} = \frac{1}{\tau} c_0\, e^{-i2\pi\frac{t_0}{\tau}\ell} + \frac{1}{\tau} c_1\, e^{-i2\pi\frac{t_1}{\tau}\ell} + \dots$$

$$d_\ell * a_\ell = \overbrace{\frac{1}{\tau} c_0\, e^{-i2\pi\frac{t_0}{\tau}\ell} * \underset{\sim}{a_\ell}}^{=0} + \overbrace{\frac{1}{\tau} c_1\, e^{-i2\pi\frac{t_1}{\tau}\ell} * a_\ell}^{=0} + \dots$$

conv. of .. $\left(\delta[n] - e^{-i2\pi\frac{t_k}{\tau}}\delta[n-1]\right)$

$$\boxed{\underline{d_\ell} * a_\ell = 0}$$

↑

known

$$a_l * d_l = \sum_{l=0}^{K} a_l \, d_{n-l} = 0$$

write in matrix-vector form

$$\begin{array}{r} n=0 \rightarrow \\ n=1 \rightarrow \\ \vdots \\ n=K \rightarrow \end{array} \underbrace{\begin{bmatrix} d_0 & d_{-1} & \cdots & d_{-K} \\ d_1 & d_0 & \cdots & d_{-K+1} \\ \vdots & & & \\ d_K & d_{K-1} & \cdots & d_0 \end{bmatrix}}_{:= S} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix}}_{a} = 0$$

2K+1 Fourier series coefficients

S is $(K+1) \times (K+1)$

Summary of the algorithm

1. Determine the Fourier series coefficients d_n

2. Form the matrix $(k+1)\times(k+1)$ matrix S

3. Solve $Sa = 0 \Rightarrow a_0, a_1 \ldots, a_k$

4. $$A(z) = \sum_{m=0}^{k} a_m z^{-m} = \prod_{\ell=0}^{k-1} (1 - \alpha_\ell z^{-1}) \quad \text{TBD}$$

$$\alpha_\ell = u_\ell = e^{-i 2\pi \frac{t_\ell}{\tau}}$$

5. $\arg(u_\ell) = -2\pi \frac{t_\ell}{\tau} \Rightarrow t_\ell$!

6. Finding the c_ℓ

$$d_n = \frac{1}{\tau} \sum_{\ell=0}^{k-1} c_\ell u_\ell^n, \quad u_\ell = e^{-i 2\pi \frac{t_\ell}{\tau}}$$

$$\frac{1}{\tau} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & \cdots & u_{k-1} \\ u_0^2 & u_1^2 & \cdots & u_{k-1}^2 \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{k \times k} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \end{bmatrix}$$

solve this lin. system $\rightarrow c_z$

4.1.3. Uniqueness

$$S = \frac{1}{L} \sum_{z=0}^{k-1} c_z \begin{bmatrix} 1 & u_z^{-1} & \cdots & u_z^{-k} \\ u_z & & & \vdots \\ u_z^2 & & & \vdots \\ \vdots & & & \\ u_z^k & & & \end{bmatrix}$$

$$= \frac{1}{L} \sum_{z=0}^{k-1} c_z \begin{bmatrix} 1 \\ u_z \\ u_z^2 \\ \vdots \\ u_z^k \end{bmatrix} \begin{bmatrix} 1 & u_z^{-1} & u_z^{-2} & \cdots & u_z^{-k} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ u_0 & u_1 & \cdots & u_{K-1} & u_K \\ u_0^2 & u_1^2 & \cdots & u_{K-1}^2 & u_K^2 \\ \vdots & \vdots & & \vdots & \vdots \\ u_0^K & u_1^K & \cdots & u_{K-1}^K & u_K^K \end{bmatrix}}_{(K+1)\times(K+1) \text{ matrix}} \begin{bmatrix} 1 & \cdots \\ u_0^{-1} & \\ u_0^{-2} & \\ \vdots & \end{bmatrix}$$

Vandermonde

$$u_k = e^{-i2\pi\frac{t_k}{\tau}}$$

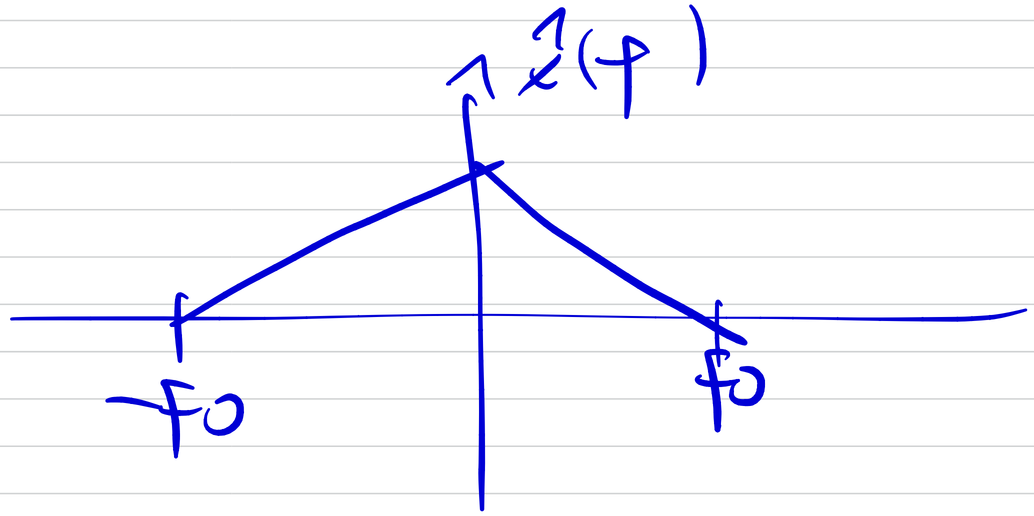
We derived an algorithm that uniquely recovers

$$x(t) = \sum_{k=0}^{K-1} c_k\, \delta(t-t_k)$$

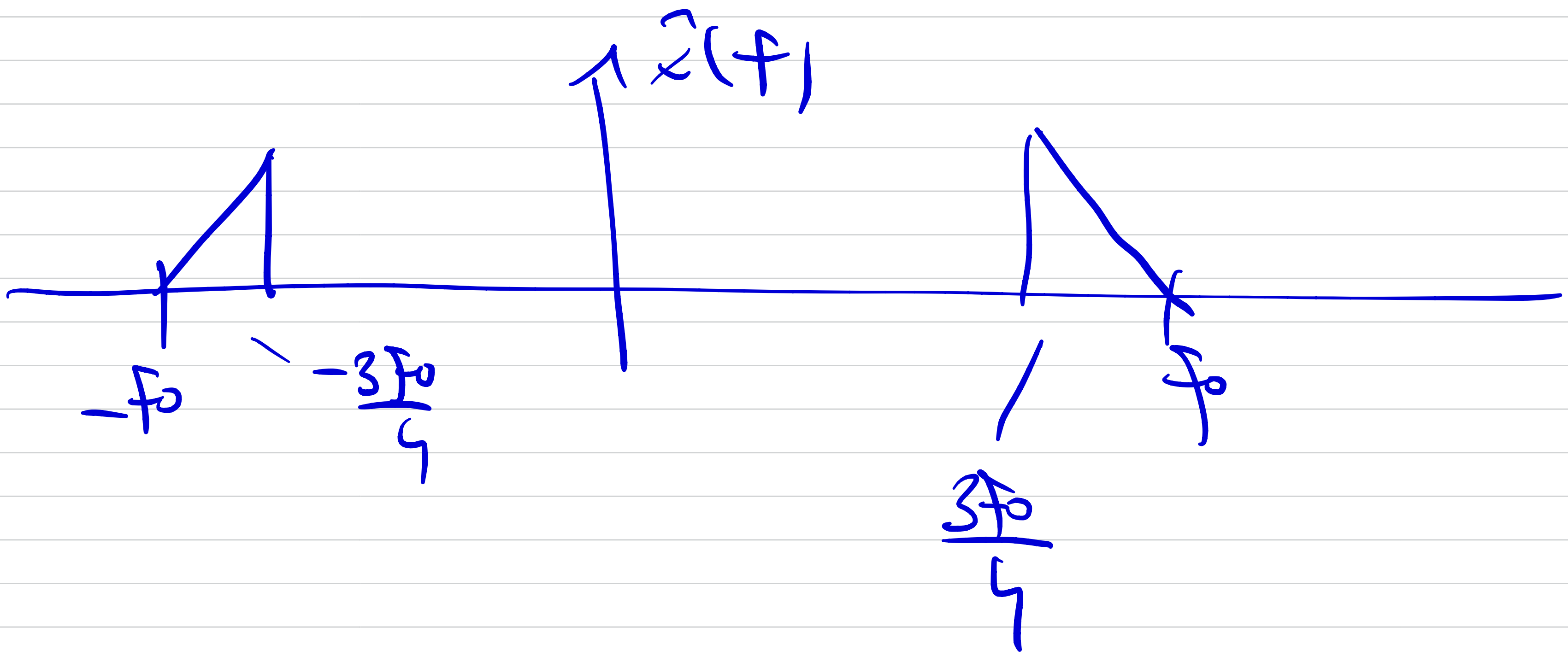
From $2K+1$ Fourier series coefficients of $x(t)$.

$$\boxed{s = 2K, \; m = 2K+1}$$

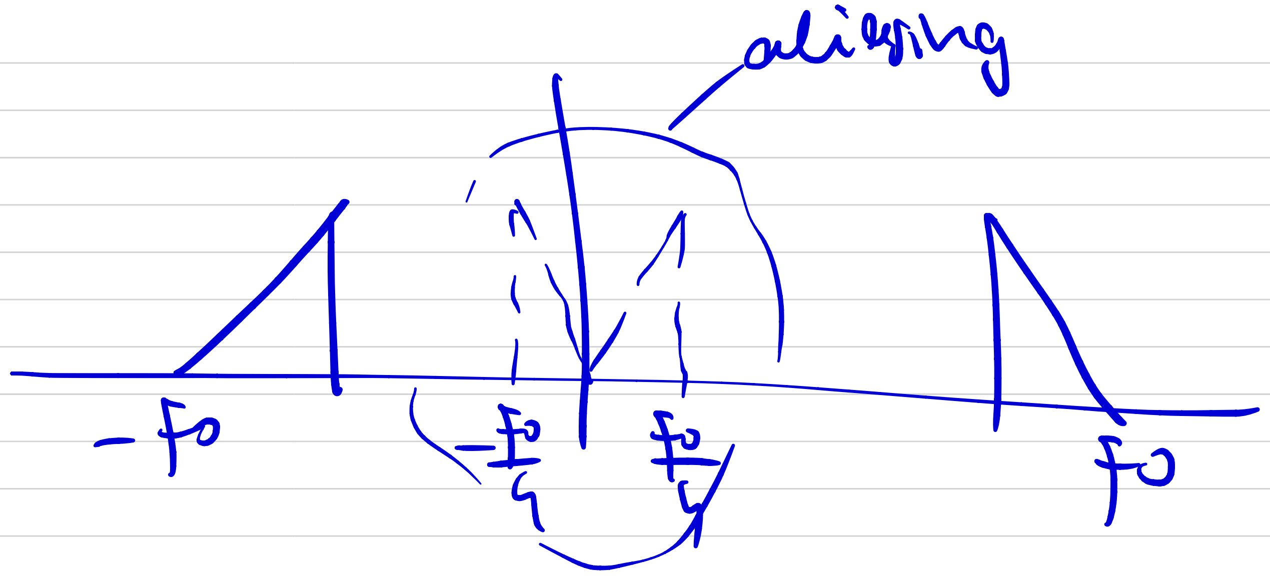
Chapter 5 : Sampling of multiband signals



sampling th. says that $f_s \geq 2f_0$ allows perfect recovery of $x(t)$ from its samples $x\left(n \cdot \underbrace{\frac{1}{f_s}}_{T_s}\right)$.



$\frac{f_0}{4} \cdot 2$



$f_s = f_0$

$$f_s = \frac{f_0}{2}$$

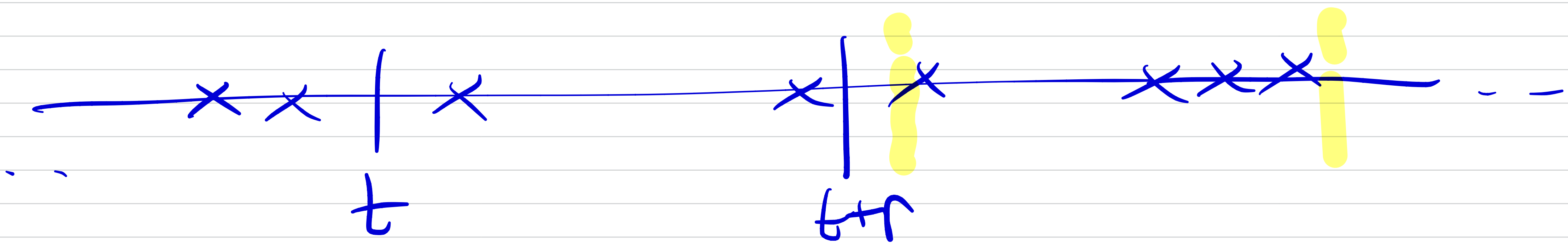
Sampling set $P = \{t_n\}$, i.e., we are given the signal values $\{x(t_n)\}$, can we recover $x(t)$ from $\{x(t_n)\}$, $t_n \in P$, and if so, what is the minimum sampling rate needed?

$$P = \{t_n\}$$

Theorem 5.1 (Landau, 1967). To reconstruct stably, we need

$$D^-(P) = \lim_{r\to\infty} \inf_{t\in\mathbb{R}} \frac{|P \cap [t, t+r]|}{r} \geq |I|,$$

where $D^-(P)$ denotes the lower Beurling density and $|I|$ is the total spectral occupancy.

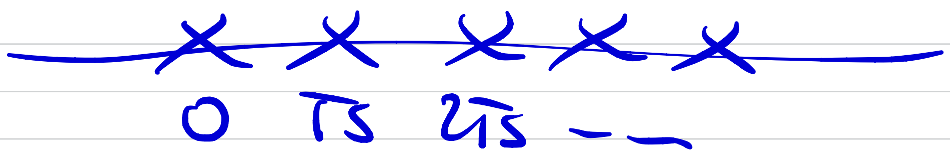


1. Fix r.

2. Slide the window of length r from $-\infty$ to ∞ and record the no. of sampling points in P that land inside the window, take the inf across t.
Note that the result depends on r!

3. Take window length to infinity and compute the limit of $\frac{|P \cap [t, t+r]|}{r}$ as $r \to \infty$.

Classical sampling rate:



interval of length r $\Rightarrow$ no. of sampling points in interval is

$$\frac{r}{T_s} = r f_s$$

divide by r $\Rightarrow$ $f_s \geq |I|$

5.2.2. Stable sampling

Def. A set of points $P=\{t_n\}$ is called a stable sampling set if for all $x_1, x_2 \in \mathcal{H}$, we have

$$A\|x_1-x_2\|_{\mathcal{H}}^2 \le \|x_1(P)-x_2(P)\|_2^2 \le B\|x_1-x_2\|_{\mathcal{H}}^2$$

for some $A>0$ and $B<\infty$.

$$\{x_1(t_n)\}_{P=\{t_n\}}$$

lower bound guarantees uniqueness

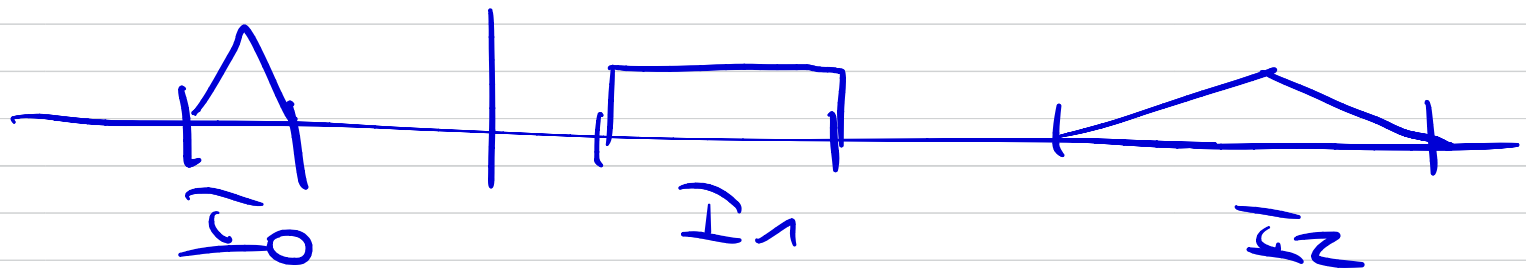
$\mathcal{H}$ is a vector space: define $T : x \to \{x(P)\}$

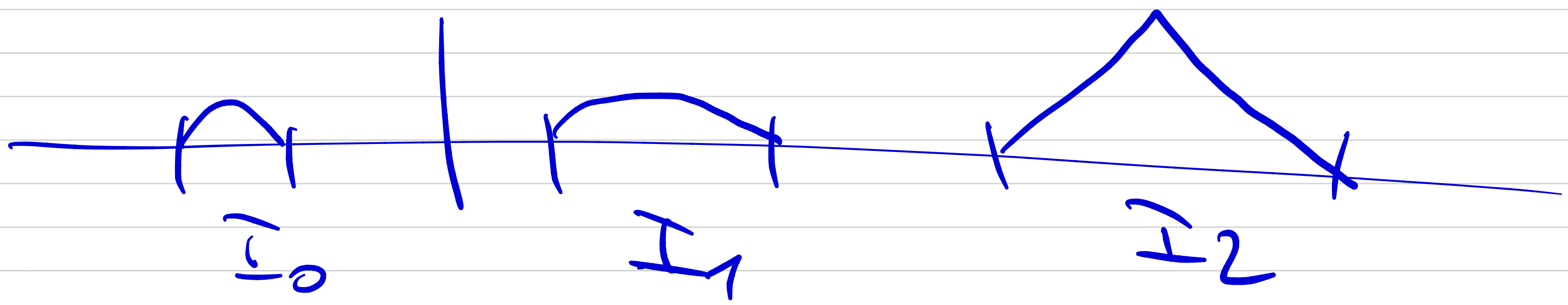
$$A\|x\|_{\mathcal{H}}^2 \le \|Tx\|_2^2 \le B\|x\|_{\mathcal{H}}^2$$

$x = x_1 - x_2$

Frame condition

Known support set : Fix support set in the frequency domain





if spectral support set is not known, then we are no longer dealing with a vector space !