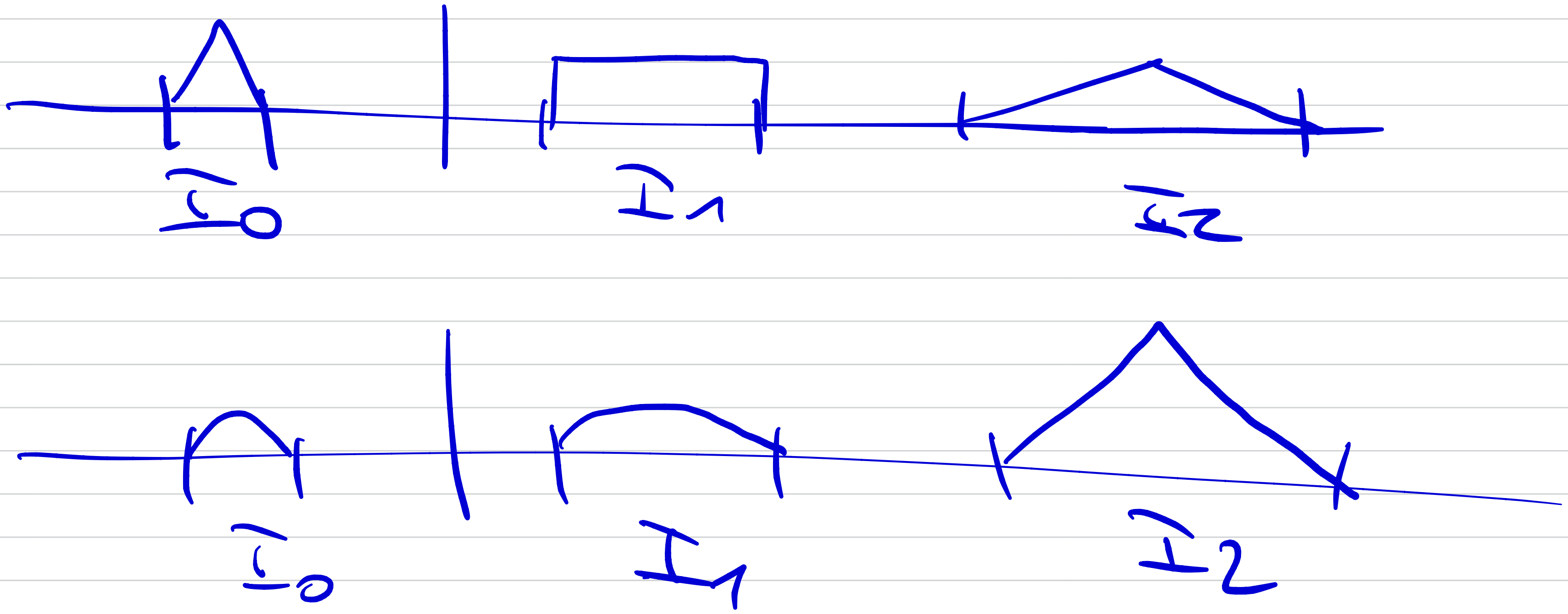
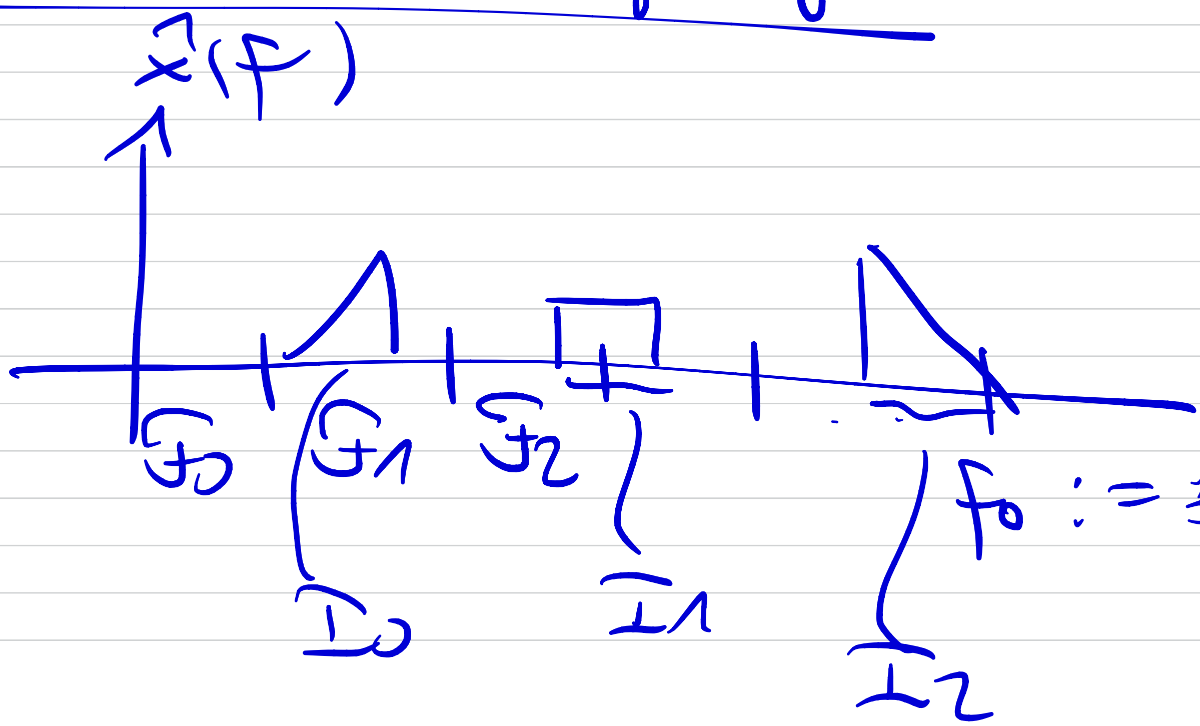
Known support set: Fix support set in the frequency domain



if spectral support set is not known, then we are no longer dealing with a vector space!

5.3. Multicoset sampling



$f_0 := \frac{1}{T}$

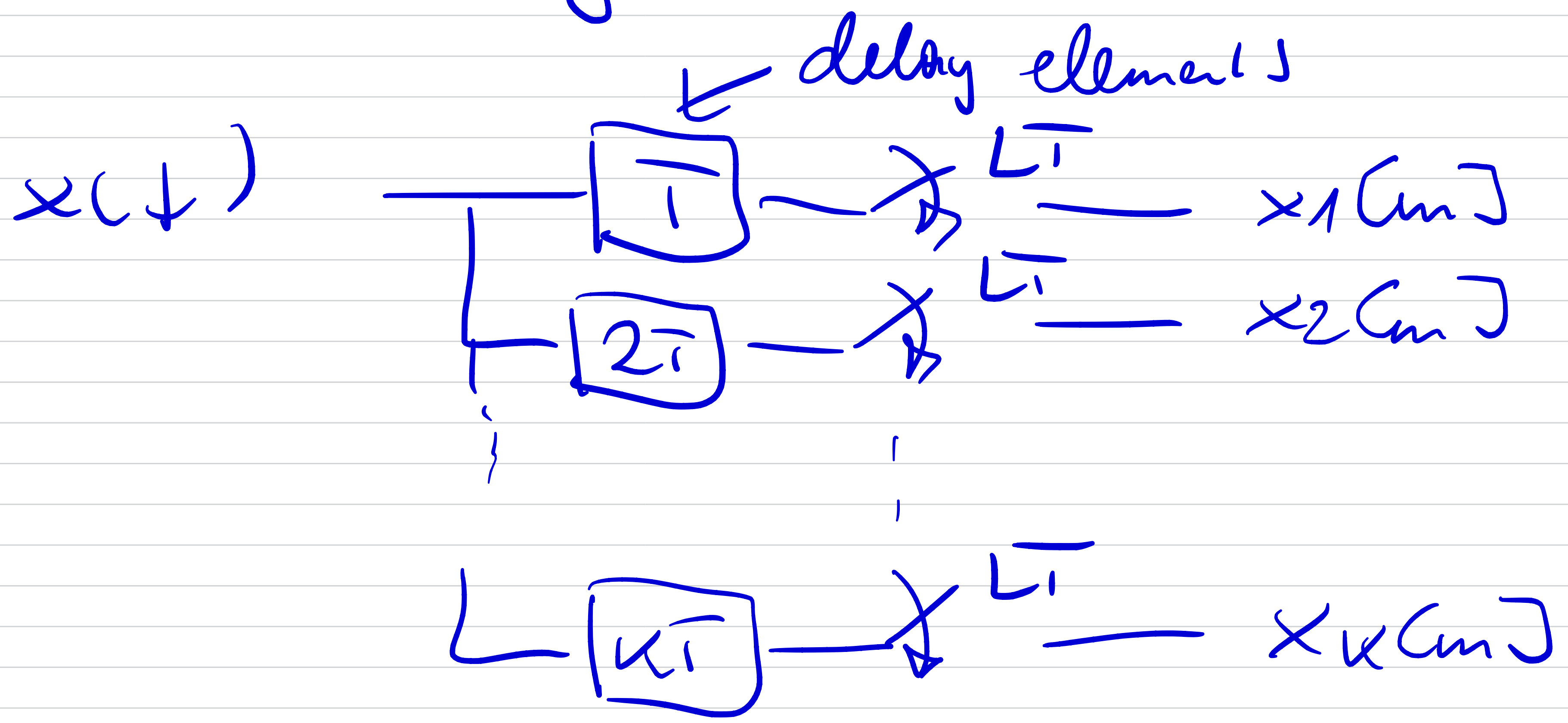
L pieces, $\frac{f_0}{L} = \frac{1}{LT}$

I ... spectral support set

$I = I_0 \cup I_1 \cup I_2$

$$|\mathcal{I}| \approx \frac{s f_0}{L}$$

s = no. of chunks that contain nonzero signal components



$$x_\ell[n] = x((nL+\ell)T), \quad n \in \mathbb{Z}$$

$$D^-(P) = \frac{K}{LT} = \frac{K}{L} f_0, \qquad |\mathcal{I}| \approx \frac{s f_0}{L}$$

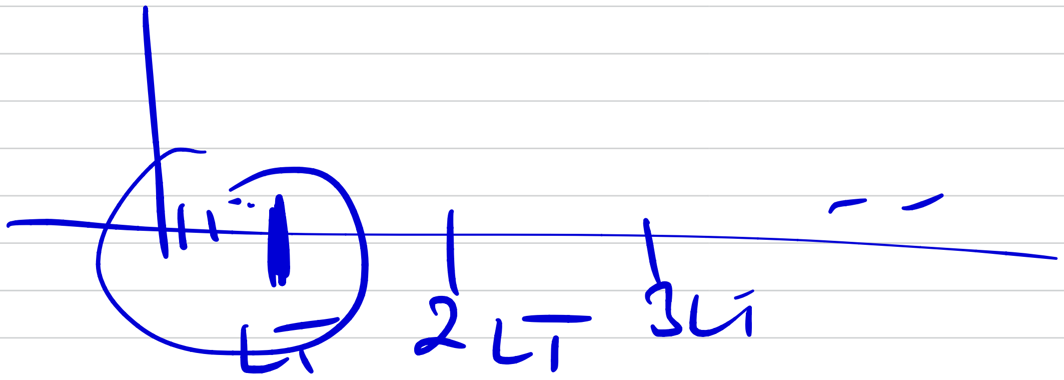
$K \le L$

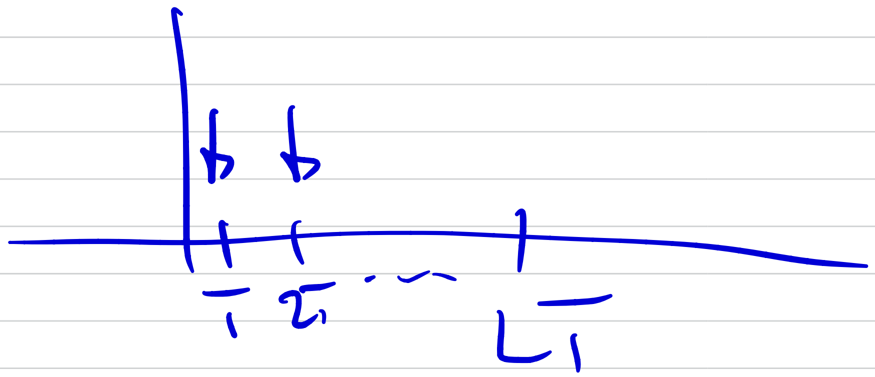
at Landau rate $D^-(P) \approx |\mathcal{I}| = \frac{s f_0}{L}$

critical sampling : $K = S$

$\uparrow$

Landau rate





$$x_d^{(s)}(f) = \sum_{m \in \mathbb{Z}} x_s[m]\, e^{-i2\pi f m \overbrace{TL}^{1/(\text{sampling rate})}}$$

(DTFT of $x_s[m]$)

$$= \sum_{m \in \mathbb{Z}} x\big((mL+s)T\big)\, e^{-i2\pi f m T L}$$

$$= e^{i2\pi f sT} \sum_{m \in \mathbb{Z}} x\big((mL+s)T\big)\, e^{-i2\pi f (mTL + sT)}$$

$$= e^{i2\pi f \ell T} \frac{1}{LT} \sum_{m\in\mathbb{Z}} \hat{x}\left(f+\frac{m}{TL}\right) e^{i2\pi \frac{m}{L\cancel{T}} \ell \cancel{T}}$$

(Poisson summation formula

$$\sum_{\ell\in\mathbb{Z}} s(t+\ell T) = \overbrace{\frac{1}{T} \sum_{\ell\in\mathbb{Z}} \hat{s}\left(\frac{\ell}{T}\right) e^{i2\pi \frac{\ell}{T} t}}^{\text{Fourier series}}$$
)

$$= e^{i2\pi f \ell T} \frac{1}{LT} \sum_{m\in\mathbb{Z}} \hat{x}\left(f+\frac{m}{TL}\right) e^{i2\pi \frac{m}{L} \ell}, \quad f\in[0,1)$$

$$v_\ell(f) := x_d^{(\ell)}(f)\, e^{-i2\pi f \ell T} LT = \sum_{m\in\mathbb{Z}} \hat{x}\left(f+\frac{m}{TL}\right) e^{i2\pi \frac{m}{L}\ell}$$

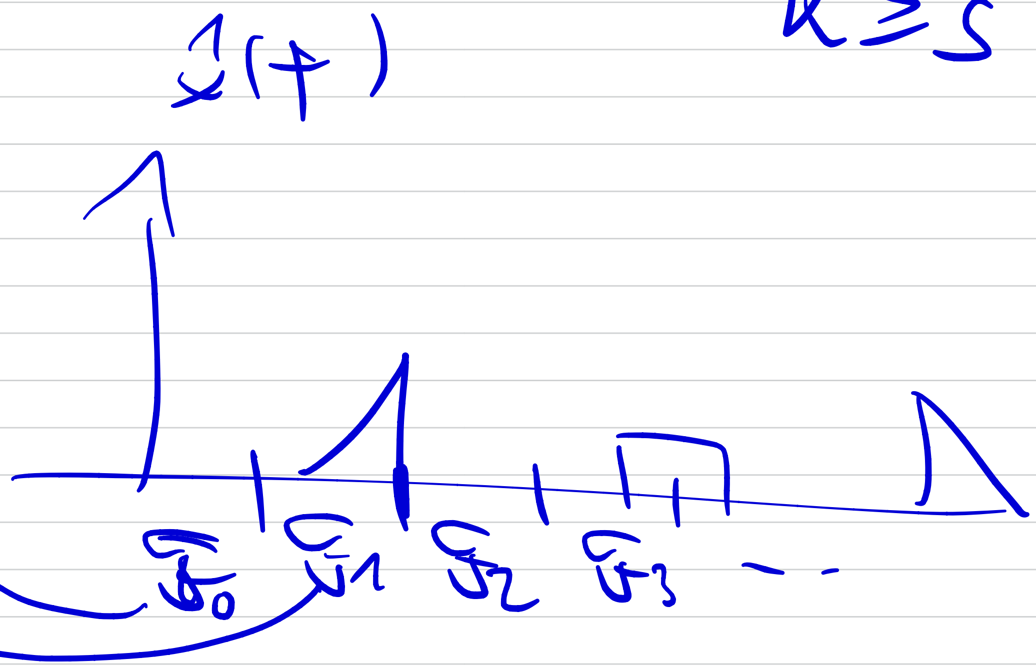
$$f\in\left[0, \frac{1}{LT}\right)$$

$$\underbrace{\begin{bmatrix} v_1(f) \\ v_2(f) \\ \vdots \\ v_K(f)\end{bmatrix}}_{\substack{K\times 1 \\ v(f)}} = \underbrace{\underbrace{\begin{bmatrix} 1 & e^{i2\pi\frac{1}{L}} & e^{i2\pi\frac{2}{L}} & \cdots \\ 1 & e^{i2\pi \frac{2}{L}} & \cdots & \\ \vdots & & & \end{bmatrix}}_{K\times L}}_{A} \underbrace{\underbrace{\begin{bmatrix} x(f)\\ x(f+\frac{1}{LT}) \\ \vdots \\ x(f+\frac{L-1}{LT})\end{bmatrix}}_{L\times 1}}_{x(f)}, \quad f\in[0,\tfrac{1}{LT})$$

$K \leq L$

$K \geq S$

$$f\in[0,\tfrac{1}{LT}) \quad \begin{bmatrix} x(f) \\ \hline x(f+\frac{1}{LT}) \\ \vdots \\ \\ \end{bmatrix} \begin{matrix} \leftarrow f_0 \\ \leftarrow f_1 \\ \\ \\ \end{matrix}$$



Summary: 1. $v(f) = A\,\vec{x}(f) = A_\Lambda \vec{x}_\Lambda(f)$

2. $m = s \Rightarrow A_\Lambda$ is $s \times s$ & regular

3. $\hat{x}_\Lambda(f) = A_\Lambda^{-1} v(f)$

4. $\hat{x}_\Lambda(f) \longrightarrow \vec{x}(f)$

5. organize $\underbrace{\vec{x}(f)}_{\text{vector}}$, $f \in [0, \frac{1}{L})$ into spectrum $\vec{x}(f)$

5.4. Spectrum-Blind Sampling

Consider the support set Λ to be unknown, but we do know that $|\Lambda| \leq s$

$$\mathcal{X}(C) = \bigcup_{|\mathcal{I}| \leq C} \mathcal{B}(\mathcal{I})$$

This set does not form a vector space.

Stable sampling

$$A\|x_1-x_2\|^2 \le \|Tx_1 - Tx_2\|_2^2 \le B\|x_1-x_2\|^2, \quad \forall x_1, x_2 \in \mathcal{X}(C)$$

Note that for $x_1, x_2 \in \mathcal{X}(C)$, $x_1 - x_2 \notin \mathcal{X}(C)$, but $x_1 - x_2 \in \mathcal{X}(2C)$

$$A\|x\|^2 \le \|Tx\|_2^2 \le B\|x\|^2, \quad \forall x \in \mathcal{X}(2C)$$

$$\mathcal{D}^-(\rho) \ge 2C, \quad \text{necessary}$$

How about sufficiency?

$$\|\hat{x}_1(f) - \hat{x}_2(f)\|_0 \le 2s \le k, \quad \forall f \in [0, \tfrac{1}{L_1})$$

$$|I| = \frac{s}{L_1} \le \frac{1}{L_1}\frac{k}{2} = \frac{1}{2}\underbrace{\frac{k}{L_1}}_{\mathcal{D}^-(\rho)} = \frac{\mathcal{D}^-(\rho)}{2}$$

$$\mathcal{D}^-(P) \geq 2|\mathcal{I}|$$

$|\mathcal{I}| \leq C \implies \mathcal{D}^-(P) \geq 2C$ is sufficient for stable sampling

(MMV + subspace alg. $\Rightarrow$ recovery at Landau rate

Heckel & B, 2013, Identification of sparse linear operators)

Chapter 6 : The ESPRIT Algorithm

Paulraj, Roy, Kailath, 1993 Asilomar

Sampling of signals with FRI : $x(t) = \sum_{\ell=0}^{K-1} c_\ell \delta(t - t_\ell)$

$$\boxed{d_n = \frac{1}{\tau} \sum_{\ell=0}^{K-1} c_\ell e^{-i2\pi n \frac{t_\ell}{\tau}}, \quad n \in \mathbb{Z}}$$

Fourier series coefficients of

$$x_n = \sum_{k=1}^{K} \alpha_k z_k^n , \qquad z_k = e^{-d_k} e^{i 2\pi f_k} , \; k = 1, \dots, K$$

$d_k \geq 0$ ... damping factor

f_k ... frequency

$N \geq 2K$ ... no. of measurements

FRI recovered for $z_k = e^{-i 2\pi \frac{t_k}{\tau}}$

$$V_{L \times K} := \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & & z_K \\ z_1^2 & z_2^2 & & z_K^2 \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & & z_K^{L-1} \end{pmatrix}}_{L \times K} , \text{ Vandermonde matrix}$$

$L \geq K$

nodes z_k

Hankel matrix

$$\mathcal{H}_L(x_0, x_1, \ldots, x_{N-1}) := \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-L-1} & x_{N-L} \\ x_1 & x_2 & \cdots & x_{N-L} & x_{N-L+1} \\ \vdots & & & & \\ \vdots & & & & \\ x_{L-1} & x_L & \cdots & x_{N-2} & x_{N-1} \end{pmatrix}$$

L ... parameter, it controls the aspect ratio of $\mathcal{H}_L$

6.2 Signal and Noise Subspaces

$$X = \mathcal{H}_L(x_0, x_1, \ldots, x_{N-1})$$

$$= V_L D_\alpha \overline{V_{N-L+1}}^T \in \mathbb{C}^{L \times (N-L+1)}$$

$$V_L D_\alpha V_{N-L+1}^T =$$

$$= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_K \\ z_1^2 & z_2^2 & \cdots & z_K^2 \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & \cdots & z_K^{L-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_K \end{pmatrix} \begin{pmatrix} 1 & z_1 & \cdots & z_1^{N-L} \\ 1 & z_2 & \cdots & z_2^{N-L} \\ \vdots & \vdots & & \\ 1 & z_K & \cdots & z_K^{N-L} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_K \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & \cdots & z_K^{L-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_1 z_1 & \cdots & \alpha_1 z_1^{N-L} \\ \alpha_2 & \alpha_2 z_2 & \cdots & \alpha_2 z_2^{N-L} \\ \vdots & \vdots & & \vdots \\ \alpha_K & \alpha_K z_K & \cdots & \alpha_K z_K^{N-L} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{K} \alpha_k & \sum_{k=1}^{K} \alpha_k z_k & \cdots & \sum_{k=1}^{K} \alpha_k z_k^{N-L} \\ \sum_{k=1}^{K} \alpha_k z_k & \sum_{k=1}^{K} \alpha_k z_k^2 & & \\ \vdots & \vdots & & \\ \sum_{k=1}^{K} \alpha_k z_k^{L-1} & & & \end{pmatrix}$$

$$= \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-L} \\ x_1 & x_2 & \cdots & \\ \vdots & & & \\ x_{L-1} & \cdots & \cdots & \end{pmatrix} = X$$

$$x_n = \sum_{k=1}^{K} \alpha_k z_k^n$$

$$V_L = V_{L\times K}(z_1, z_2, \ldots, z_K) \in \mathbb{C}^{L\times K}$$

$$D_\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_K) \in \mathbb{C}^{K\times K}$$

$$V_{N-L+1} = V_{(N-L+1)\times K}(z_1, z_2, \ldots, z_K) \in \mathbb{C}^{(N-L+1)\times K}$$

$$x_n = \sum_{k=1}^{K} \alpha_k z_k^n$$

$$N-K \geq L+1 \geq L-1$$

L : $\quad K+1 \leq L \leq N-K-1$

$$N-L+1 \geq K$$

$$N-K \geq L-1$$

dimension

$$r(A)+r(B)-L \leq r(AB) \leq \min(r(A), r(B))$$

$$\underbrace{V_L D_\alpha}_{L\times K \quad K\times K} \qquad 2K-K \leq r(V_L D_\alpha) \leq K$$

$$r(V_L D_\alpha) = K$$

$$\operatorname{rank}(X) = \operatorname{rank}(V_L D_\alpha V_{N-L+1}^T) = K$$

SVD of X

$$X = \underbrace{\begin{pmatrix} \overset{L\times K}{S} & S_\perp \end{pmatrix}}_{U \in \mathbb{C}^{L\times L}} \underbrace{\overbrace{\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}}^{L\times(N-L+1)}}_{\Sigma} \underbrace{\overbrace{\begin{pmatrix} R^H \\ R_\perp^H \end{pmatrix}}^{(N-L+1)\times(N-L+1)}}_{W^H} = S \Delta R^H$$

$$\Delta = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_K)$$

$$X = S \Delta R^H$$

6.3 The ESPRIT Algorithm

$$X = V_L D_\alpha V_{N-L+1}^T$$

$\downarrow$

Vandermonde

$V_\downarrow \in \mathbb{C}^{(L-1)\times k}$ ... top $(L-1)$ rows of V_L

$V_\uparrow \in \mathbb{C}^{(L-1)\times k}$ — bottom $(L-1)$ rows of V_L

$V_\uparrow = V_\downarrow D_z$

$$V_L = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ z_1^2 & z_2^2 & & z_k^2 \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & & z_k^{L-1} \end{pmatrix}$$

$$D_z = \begin{pmatrix} z_1 & & \emptyset \\ & z_2 & \\ \emptyset & & \ddots \\ & & & z_k \end{pmatrix}$$

shift property of $\overline{F_l}$

columns of both S and V_L form ONBs for signal subspace S

$\Rightarrow \exists$ invertible P s.t. , $P \in \mathbb{C}^{k\times k}$

$$S = V_L P$$

$$[s_1\ s_2 \ldots s_k] = [v_1\ v_2 \ldots v_k] \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & \cdots & \\ \vdots & & \end{pmatrix}$$

$$S_1 = \sum_{i=1}^{K} v_i \, p_{i,1}$$

$S_\downarrow$ -- top $(L-1)$ rows of S

$S_\uparrow$ -- bottom $(L-1)$ rows of S

$$S = V_L P \Rightarrow \begin{aligned} S_\downarrow &= V_\downarrow P \\ S_\uparrow &= V_\uparrow P \end{aligned}$$

$$V_\uparrow = V_\downarrow D_z$$

$$S_\downarrow = V_\downarrow P \Rightarrow V_\downarrow = S_\downarrow P^{-1}$$

$$S_\uparrow = V_\uparrow P \Rightarrow V_\uparrow = S_\uparrow P^{-1}$$

$$S_\uparrow P^{-1} = S_\downarrow P^{-1} D_z \quad | \cdot P$$

$$S_\uparrow = S_\downarrow \underbrace{P^{-1} D_z P}_{\Phi}$$

Moore-Penrose pseudo-inverse

$$\underline{S_{\uparrow}} = \underline{S_{\downarrow} \Phi} \Rightarrow \underline{\Phi} = S_{\downarrow}^{\dagger} S_{\uparrow}$$

the eigenvalues of $\underline{\Phi}$ are equal to the $\underbrace{\text{eigenvalues of } D_z}_{z_1, z_2, \ldots, z_K}$

similarity principle

1. $x_0, x_1, \ldots, x_{N-1}$ data $\Rightarrow$ Hankel matrix X
2. SVD of X $\Rightarrow$ $\underbrace{[\,S \;\; S_{\perp}\,]}_{U} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} R^H \\ R_{\perp}^H \end{bmatrix}$

   S ... first L columns of U

3. compute $\underline{\Phi} = S_{\downarrow}^{\dagger} S_{\uparrow}$
4. $\operatorname{eig}(\underline{\Phi}) = z_1, z_2, \ldots, z_K$
5. α_k by solving a linear system of equations

Finalizing that this alg. delivers the correct solution

1. $\Phi = S_\downarrow^+ S_\uparrow$ is, indeed, a solution to $\underline{S_\uparrow = S_\downarrow Y}$

2. solution is unique

$$S_\downarrow = \overbrace{V_{L-1}}^{V_\downarrow} P$$

$$S_\uparrow = \underbrace{V_{L-1} D_Z}_{V_\uparrow} P$$

$$S_\downarrow \underbrace{S_\downarrow^+ S_\uparrow}_{\Phi} = V_{L-1} \underbrace{P P^{-1}}_{I} V_{L-1}^+ V_{L-1} D_Z P$$

$$= \underbrace{V_{L-1} V_{L-1}^+}\, V_{L-1} D_Z P = V_{L-1} D_Z P = S_\uparrow$$

$$S_\downarrow \Phi = S_\uparrow$$

$$S_\uparrow = S_\flat \overline{\underline{\Phi}}$$

$$S_\flat = V_{L-1} P, \quad r(V_{L-1}) = K, \quad r(P) = K \Rightarrow r(S_\flat) = K$$

Similarity principle.

Def. 6.1. The matrices $X \in \mathbb{C}^{n\times n}$ and $Y \in \mathbb{C}^{n\times n}$ are similar if $\exists$ $P \in \mathbb{C}^{n\times n}$, P invertible, s.t.

$$X = P^{-1} Y P.$$

Th. 6.2. Let A and B be similar matrices. Then, A and B have the same eigenvalues (with the same geometric multiplicities).

Proof. $$A = P^{-1} B P \Rightarrow B = P A P^{-1}$$

if $\underline{A}u = \underline{\lambda} u$, then $P^{-1}BPu = \lambda u \Rightarrow B\underbrace{Pu}_{u'} = \lambda \underbrace{Pu}_{u'}$

missing in notes

$$\underline{B}u' = \underline{\lambda} u'$$

$$Bu = \lambda u$$

$$PAP^{-1}u = \lambda u$$

$$A\underbrace{P^{-1}u}_{u''} = \lambda \underbrace{P^{-1}u}_{u''} \Rightarrow$$

if λ is an eigenvalue of B, then λ is also an eigenvalue of A