if $\underline{A}u = \underline{\lambda}u$, then $P^{-1}BPu = \lambda u \Rightarrow B\underbrace{Pu}_{u'} = \lambda \underbrace{Pu}_{u'}$

missing in notes

$\underline{B}u' = \underline{\lambda}u'$

$Bu = \lambda u$

$PAP^{-1}u = \lambda u$

$A\underbrace{P^{-1}u}_{u''} = \lambda \underbrace{P^{-1}u}_{u''} \Rightarrow$ if λ is an eigenvalue of B, then λ is also an eigenvalue of A

6.4. Finding the Zeros of a Polynomial

$p(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_k z^k$

all zeros have multiplicity one

1. $p(z) = (\alpha_0 \; \alpha_1 \; \cdots \; \alpha_k) \begin{pmatrix} 1 \\ z \\ \vdots \\ z^k \end{pmatrix}$

2. $p(z_i) = 0$

$$\underbrace{(\alpha_0 \; \alpha_1 \; \cdots \; \alpha_k)}_{\alpha} \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & & z_{k-1} \\ \vdots & \vdots & & \vdots \\ z_0^k & z_1^k & & z_{k-1}^k \end{pmatrix}}_{= V \in \mathbb{C}^{(k+1) \times k}} = \underbrace{(0 \; 0 \cdots 0)}_{\overline{0}}$$

$$\alpha V = 0$$

Conceptual insight: V is a basis for the right-null-space of α

3. SVD

$$\alpha = \underbrace{[u_1 \;\; u_2]}_{1\times(k+1)} \underbrace{\begin{bmatrix} \overset{1\times 1}{\sigma} & 0_{1\times k} \\ 0_{k\times 1} & 0_{k\times k} \end{bmatrix}}_{(k+1)\times(k+1)} \underbrace{\begin{bmatrix} w^T \\ Z^T \end{bmatrix}}_{(k+1)\times(k+1)}$$

w: $(k+1)\times 1$, $\underbrace{(k+1)\times k}_{Z}$, $[u_1]$: 1×1

$$= u_1 \sigma w^T$$

w is orthonormal to the columns of Z

$$\alpha Z = u_1 \sigma \underbrace{w^T Z}_{=0} = 0$$

$$\Updownarrow$$

$$(\alpha V = 0)$$

4. As both V and Z are bases for the right-null-space of α, they must be related through a transformation matrix T according to

$$\overset{(k+1)\times k}{V} = Z \overset{k\times k}{T}$$

$(K+1)\times K$

Shift-property

$$V_\uparrow = V_\downarrow D_z$$

$$V = ZT$$

$$\left.\begin{aligned} V_\uparrow &= Z_\uparrow T \\ V_\downarrow &= Z_\downarrow T \end{aligned}\right\} = Z_\uparrow T = Z_\downarrow T D_z$$

$$Z_\uparrow = Z_\downarrow \underbrace{T D_z T^{-1}}_{\underline{\Phi}} = Z_\downarrow \underline{\Phi}$$

$$\underline{\Phi} = Z_\downarrow^{+} Z_\uparrow$$

eigenvalues of $\underline{\Phi}$ = eigenvalues of D_z = zeros of polynomial

Chapter 7. The restricted isometry property

1. $m \propto s^2$ (prop. to)

m ... no. of measurements
s ... sparsity level

2. $m \propto s \log n$

3. $m \propto s \quad (2s, s+1, s)$

FRI

spectrum-blind sampling

uniqueness

how many s-dim. subspaces are there in n-dim. ambient space?

$$\left[\binom{n}{s} \approx \left(\frac{n}{s}\right)^s \right]$$

Restricted isometry property (proof is mechanical)

Def. 7.1. For each integer $s = 1, 2, \ldots$, define the isometry constant δ_s of a matrix Φ as the smallest number such that

$$(1-\delta_s)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta_s)\|x\|_2^2$$

holds for all s-sparse vectors.

Theorem 7.2. Let $y = \Phi x$. Assume that $\delta_{2s} < \sqrt{2}-1$. Then, the solution x^* to

$$\min_{\tilde{x} \in \mathbb{R}^n} \|\tilde{x}\|_1 \qquad \text{subject to} \quad \Phi \tilde{x} = y$$

obeys

$$\|x^* - x\|_1 \leq C_0 \|x - x_s\|_1,$$

where x_s is the vector obtained by setting all but the s largest (in absolute value) entries of x equal to zero.

Theorem 7.3. Let $y = \Phi x + n$. Assume that $\delta_{2s} < \sqrt{2} - 1$ and $\|n\|_2 \leq \varepsilon$. Then, the solution x^* to

$$\underset{\tilde{x} \in \mathbb{R}^n}{\text{minimize}} \ \|\tilde{x}\|_1 \qquad \text{subject to} \quad \|y - \Phi \tilde{x}\|_2 \leq \varepsilon$$

obeys

$$\|\hat{x} - x\|_2 \leq C_0 s^{-1/2} \|x - x_s\|_1 + C_1 \varepsilon.$$

Chapter 8. The Johnson–Lindenstrauss Lemma

- $\mathcal{U}$ .. set of m points in $\mathbb{R}^n$
- embed these points into a lower-dimensional Euclidean space, $\mathbb{R}^k$, $k < n$
- Q: how small can we choose k as a function of m and ε?

$$k = \Theta\left(\frac{\log(m)}{\varepsilon^2} \right)$$

Lemma 8.1. (JL Lemma). Choose ε with $0<\varepsilon<1$ and suppose that k satisfies

$$k \geq \frac{8}{\varepsilon^2 - \varepsilon^3} \log(2m).$$

Then, for every set $\mathcal{U}$ of m points, there exists a map $f: \mathbb{R}^n \to \mathbb{R}^k$ such that <u>for all $u, u' \in \mathcal{U}$,</u> we have

$$(1-\varepsilon)\, \|u-u'\|_2^2 \leq \| f(u) - f(u')\|_2^2 \leq (1+\varepsilon)\, \|u-u'\|_2^2. \quad (*)$$

Concentration of measure inequality

Lemma 8.2. Let $A \in \mathbb{R}^{k \times n}$ be a random matrix with i.i.d. $\mathcal{N}(0, 1/k)$ entries. Then, for ε with $0<\varepsilon<1$ and fixed $u \in \mathbb{R}^n$,

$$P\left(\left| \|Au\|^2 - \overbrace{E[\|Au\|^2]}^{\|u\|^2} \right| \geq \varepsilon \|u\|^2 \right) < 2 e^{-k \frac{\varepsilon^2 - \varepsilon^3}{4}}$$

with

$$E[\|Au\|^2] = \|u\|^2.$$

Lemma 8.2 tells us what the probability of this not being satisfied is

$$-\varepsilon\|u\|^2 \le \|Au\|^2 - \|u\|^2 \le \varepsilon\|u\|^2$$

$$\boxed{(1-\varepsilon)\|u\|^2 \le \|Au\|^2 \le (1+\varepsilon)\|u\|^2}$$

Proof of JL Lemma based on Lemma 8.2.

$$f(u) = Au$$

\# pairs $\{u, u'\}$: $\frac{m(m-1)}{2} < m^2$

union bound argument $\Rightarrow$ $(*)$ is violated with prob. $< m^2 2e^{-k\frac{\varepsilon^2-\varepsilon^3}{4}}$

$$2m^2 e^{-k\frac{\varepsilon^2-\varepsilon^3}{4}} < 1/2 \qquad \leftarrow \text{rework}$$

arbitrary

$$k \ge \frac{4}{\varepsilon^2-\varepsilon^3} 2\log(2m). \quad \square$$

Chapter 9. Verifying the RIP through the JL Lemma

prove the RIP for random matrices

$\mathbb{R}^n$ ... ambient space

S ... sparsity pattern

$\mathcal{X}_S$ ... the set of all vectors in $\mathbb{R}^n$ that are zero outside S

approach: fix X_S & construct nets of points in $\mathcal{X}_S$, then apply JL (conc. of measure) together with union bound, count the no. of diff. X_S

Lemma 9.1. Let $\Phi \in \mathbb{R}^{m\times n}$ be an i.i.d. $\mathcal{N}(0, 1/m)$ matrix. Then, for every set S with $|S| = s < m$ and every $0<\delta<1$, we have

$$(1-\delta)\|x\| \leq \|\Phi x\| \leq (1+\delta)\|x\| \quad , \forall x \in \mathcal{X}_S$$

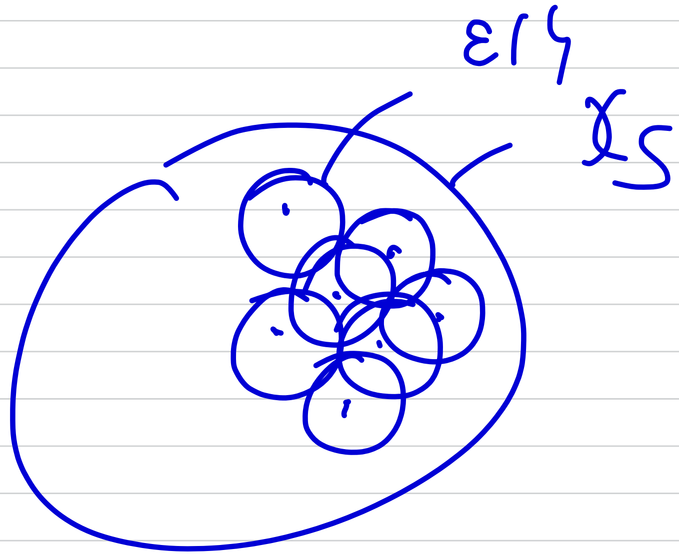
with prob.

$$\geq 1 - 2(12/\delta)^k e^{-c_0(\delta/2)m}$$

where $c_0(x) = \frac{1}{4}(x^2 - x^3)$.

Proof. $\|x\| = 1$. Choose a finite set of points Q_S s.t. i) $Q_S \subseteq X_S$, $\|q\| = 1$ for all $q \in Q_S$, and ii) for all $x \in X_S$ with $\|x\| = 1$, we have

$$\min_{q \in Q_S} \| x - q \| \leq \delta/4 .$$



From the theory of covering numbers, we get

$$|Q_S| \leq (12/\delta)^k$$

apply union bound, Lemma 1.2 (conc. of measure)

$$(1-\delta/2)\|q\|^2 \le \|\Phi q\|^2 \le (1+\delta/2)\|q\|^2, \ \forall q \in Q_S$$

holds with probability

$$\ge 1 - 2(12/\delta)^k e^{-c_0(\delta/2)m}.$$

define A as the smallest no. s.t.

$$\|\Phi x\| \le (1+A)\|x\| \le (1+\delta)\|x\|$$

$x \in X_S$, $\|x\|=1$: know that $\exists q \in Q_S$ s.t. $\|x-q\| \le \delta/4$

$$\|\Phi x\| = \|-\Phi q - \Phi x + \Phi q\|$$

$$\le \underbrace{\|\Phi q\|}_{\le 1+\delta/2} + \underbrace{\|\Phi(x-q)\|}_{\le (1+A)\underbrace{\|x-q\|}_{\delta/4}}$$

$$1+\delta/2 + (1+A)\delta/4$$

$$\cancel{1}+A \le \cancel{1}+\delta/2 + (1+A)\delta/4$$

$A' \leq \delta$. □

Final step in $m \propto s \log n$ - threshold proof

How many X_S subspaces are there in $\mathbb{R}^n$?

Theorem 9.2. Suppose that m, n, and $\delta \in (0,1)$ are given. If the pdf generating $\underline{\Phi}$ satisfies the conc. of measure inequality in Lemma 8.2, then there exist constants $c_1, c_2 > 0$ depending only on δ s.t. the RIP holds for $\underline{\Phi}$ with prescribed δ and every $s \leq c_1 m / \log(n/s)$ with prob. $\geq 1 - 2e^{-c_2 m}$.

$$s \leq c_1 \frac{m}{\log(n/s)} \Rightarrow m \geq \frac{1}{c_1} s \log(n/s)$$

$$m \propto s \log(n)$$

Proof. Know that for each s-dimensional space X_S, $\underline{\Phi}$ will fail to satisfy

$$(1-\delta)\|x\| \le \|\Phi x\| \le (1+\delta)\|x\|, \ \forall x \in \mathcal{X}_S$$

with prob. $\le 2(12/\delta)^k e^{-c_0(\delta/2)m}$

$\binom{n}{k}$ subspaces of dim. k in n-dim. ambient space

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

Apply union bound across all possibilities for $\mathcal{X}_S$ with $|S|=k$

$$\left(\frac{en}{k}\right)^k 2\left(\frac{12}{\delta}\right)^k e^{-c_0(\delta/2)m}$$

$$= 2 e^{-c_0(\delta/2)m + k[\log(en/k) + \log(12/\delta)]} \le 2 e^{-c_2 m}$$

for fixed $c_1 > 0$, whenever

$$k \le \frac{c_1 m}{\log(n/k)}$$

$$c_2 \leq \underline{c_0(\vartheta, 2) - c_1\left(1 + \frac{1+\log(12(\vartheta))}{\log(n(\ell))}\right)} . \square$$

$$m \propto s \log(n) .$$