1.3.4. Tight frames

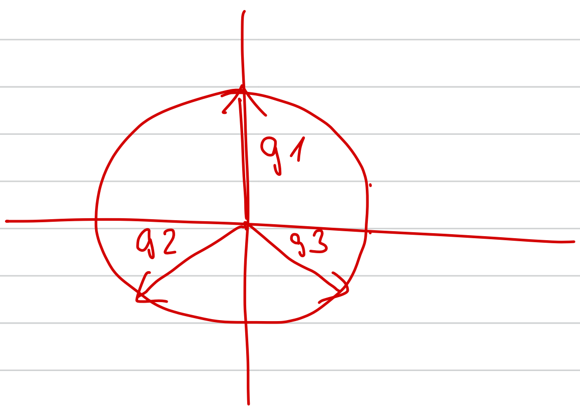
$S = AI$

Def. 1.25. A frame $\{g_k\}_{k\in K}$ with tightest possible frame bounds A, B satisfying $A = B$ is called a tight frame.

$$\langle \underset{AI}{S}x, x\rangle = \sum_k |\langle x, g_k\rangle|^2 = A\|x\|^2$$

$$\tilde{g}_k = S^{-1} g_k = \frac{1}{A} I g_k = \frac{1}{A} g_k$$

Ex. 1.27.



$$g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix} \quad g_3 = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

$$S = T^H T = \frac{3}{2} I_2, \quad A = 3/2$$

Löwdin orthogonalization

Thm. 1.28. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ with frame operator S. Denote the positive definite square root of S^{-1} by $S^{-1/2}$. Then $\{S^{-1/2} g_k\}_{k\in K}$ is a tight frame with $A = 1$, i.e.,

$$x = \sum_{k\in K} \langle x, S^{-1/2} g_k\rangle S^{-1/2} g_k, \quad \forall x \in \mathcal{H}.$$

$$\hat{x}(f) = \int x(t) e^{-i2\pi f t} dt = \langle x(\cdot), e^{i2\pi f \cdot}\rangle$$

Antoine, Berland, ...

Proof. Lemma 1.16. $\Rightarrow$ $S^{-1/2} S^{-1} = S^{-1} S^{-1/2} \mid S \cdot S$

$$S\,S^{-1/2} = S^{-1/2}\,S$$

$$x = \underbrace{S^{-1} S}_{I} x = S^{-1/2} \underbrace{S^{-1/2} S}\, x$$
$$= S^{-1/2}\, S\, S^{-1/2} x$$
$$= S^{-1/2} \sum_{k\in K} \langle S^{-1/2} x, g_k\rangle g_k$$
$$= \sum_{k\in K} \langle S^{-1/2} x, g_k\rangle S^{-1/2} g_k \quad .\square$$

Thm. 1.29. A tight frame $\{g_k\}_{k\in K}$ for $\mathcal{H}$ with $A=1$ and $\|g_k\|=1$, for all $k\in K$, is an OND for $\mathcal{H}$.

Proof.

$$\langle S g_k, g_k\rangle = A\|g_k\|^2 = 1 \quad , \forall k\in K$$
$$\langle S g_k, g_k\rangle = \sum_{j\in K} |\langle g_k, g_j\rangle|^2$$

$$S g_k = \sum_j \langle g_k, g_j\rangle g_j$$

$$= \underbrace{\|g_k\|^4}_{1} + \underbrace{\sum_{\substack{j\in K\\ j\neq k}} |\langle g_k, g_j\rangle|^2}_{=0} = 1 \quad , \forall k\in K .\square$$

Naimark

$$S = T^H T = \underbrace{[g_1 \; g_2 \dots g_N]}_{T^H} \underbrace{\begin{bmatrix} g_1^H \\ g_2^H \\ \vdots \\ g_N^H \end{bmatrix}}_{T} = A\,I = A \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \\ \vdots & & & 0 \\ & & 0 & 1 \end{bmatrix}$$

1.3.5. Exact Frames

× 1. The number of vectors in a basis is always equal to the dimension of the Hilbert space under consideration.

× 2. Every set of vectors that spans $\mathcal{H} = \mathbb{C}^M$ and has more than M vectors is necessarily redundant, i.e., the set of vectors is linearly dependent.

3. Removal of an arbitrary vector from a basis for $\mathcal{H} = \mathbb{C}^M$ leaves a set that no longer spans $\mathbb{C}^M$

4. For a given basis $\{e_k\}_{k=1}^M$ every signal $x \in \mathbb{C}^M$ has a unique representation according to

$$x = \sum_{i=1}^M \langle x, e_i \rangle \tilde{e_i}.$$

5. The basis $\{e_k\}_{k=1}^M$ and its dual $\{\tilde{e_k}\}_{k=1}^M$ satisfy

$$\langle e_k, \tilde{e}_\ell \rangle = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}.$$

Def. 1.32. Let $\{g_k\}_{k \in K}$ be a frame for $\mathcal{H}$. We call the frame $\{g_k\}_{k \in K}$ exact if, for all $m \in K$, the set $\{g_k\}_{k \neq m}$ is incomplete for $\mathcal{H}$. We call the frame $\{g_k\}_{k \in K}$ inexact if there is at least one element g_m that can be removed from the frame, so that the set $\{g_k\}_{k \neq m}$ is again a frame.

Lemma 1.33. Let $\{g_k\}_{k \in K}$ be a frame for $\mathcal{H}$ and $\{\tilde{g_k}\}_{k \in K}$ its canonical dual. For a fixed $x \in \mathcal{H}$, let $c_k = \langle x, \tilde{g_k} \rangle$ so that $x = \sum_{k \in K} c_k g_k$. If it is possible to find scalars $\{a_k\}_{k \in K}$ s.t. $\{a_k\}_{k \in K} \neq \{c_k\}_{k \in K}$ and $x = \sum_{k \in K} a_k g_k$, then we must have

$$\sum_k |a_k|^2 = \sum_k |c_k|^2 + \sum_k |c_k - a_k|^2.$$

Proof.

$$\sum_k |c_k|^2 = \sum_k a_k c_k^*$$

Lemma 1.34. Let $\{g_k\}_{k \in K}$ be a frame for $\mathcal{H}$ and $\{\tilde{g_k}\}_{k \in K}$ its canonical dual. Then, for each $m \in K$, we have

$$\sum_{k \neq m} |\langle g_m, \tilde{g_k}\rangle|^2 = \frac{1 - |\langle g_m, \tilde{g_m}\rangle|^2 - |1 - \langle g_m, \tilde{g_m}\rangle|^2}{2}.$$

Thm. 1.35. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\tilde{g_k}\}_{k\in K}$ its canonical dual. Then,

1. $\{g_k\}_{k\in K}$ is exact iff $\langle g_m, \tilde{g_m}\rangle = 1$, $\forall m \in K$
2. $\{g_k\}_{k\in K}$ is inexact iff there exists at least one $m \in K$ s.t. $\langle g_m, \tilde{g_m}\rangle \neq 1$.

Corollary 1.36. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$. If $\{g_k\}_{k\in K}$ is exact, then $\{g_k\}_{k\in K}$ and its canonical dual $\{\tilde{g_k}\}_{k\in K}$ are biorthonormal, i.e.,

$$\langle g_m, \tilde{g_k}\rangle = \begin{cases} 1, & k = m \\ 0, & \text{else} \end{cases}.$$

Conversely, if $\{g_k\}_{k\in K}$ and $\{\tilde{g_k}\}_{k\in K}$ are biorthonormal, then $\{g_k\}_{k\in K}$ is exact.

Proof. $\{g_k\}_{k\in K}$ exact $\Rightarrow \langle g_m, \tilde{g_m}\rangle = 1, \forall m \Rightarrow$ biorth.

Converse: $\{g_k\}$ and $\{\tilde{g_k}\}$ are biorth. $\Rightarrow \langle g_m, \tilde{g_m}\rangle = 1$

$\Rightarrow \{g_k\}$ is exact. $\square$

Thm. 1.37. If $\{g_k\}$ is an exact frame for $\mathcal{H}$ and $x = \sum_{k\in K} c_k g_k$ with $x \in \mathcal{H}$, then the $\{c_k\}_{k\in K}$ are unique and are given by

$$c_k = \langle x, \tilde{g_k}\rangle,$$

where $\{\tilde{g_k}\}_{k\in K}$ is the canonical dual of $\{g_k\}_{k\in K}$.

Proof. $x = \sum_{k\in K} \langle x, \tilde{g_k}\rangle g_k$

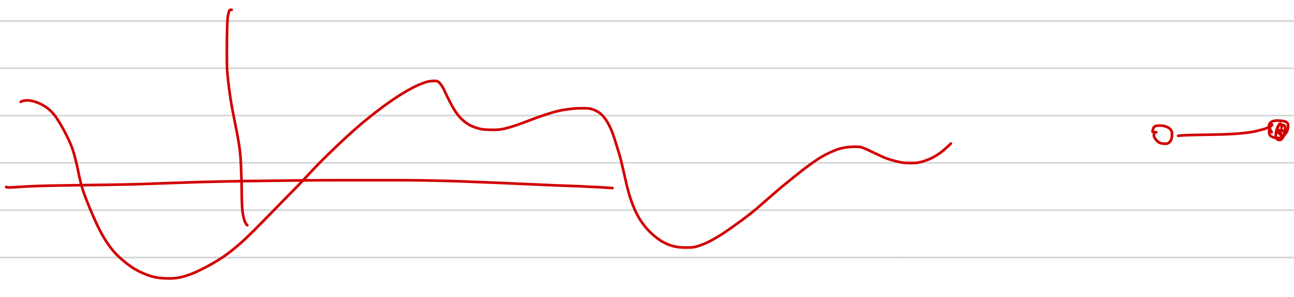
Assume that there is another set of coefficients $\{c_k\}_{k\in K}$ s.t.

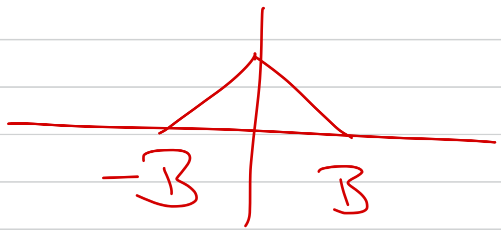
$$x = \sum_{k\in K} c_k g_k.$$

$$\langle x, \tilde{g}_m \rangle = \Big\langle \sum_{k \in K} c_k g_k, \tilde{g}_m \Big\rangle = \sum_{k \in K} c_k \underbrace{\langle g_k, \tilde{g}_m \rangle}_{= \begin{cases} 1, & k = m \\ 0, & \text{else} \end{cases}}$$

$$= c_m \quad . \quad \square$$

1.4. Sampling Theorem





$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \, dt$$

$$\{ x[k] := x(kT) \}_{k \in \mathbb{Z}} \quad \text{samples}$$

$$\hat{x}_d(f) = \sum_k x[k] e^{-i2\pi k f}$$

$$= \sum_k x(kT) e^{-i2\pi k f} \qquad x(t) e^{-i2\pi t \frac{f}{T}} \Big|_{t = kT} = x(kT) e^{-i2\pi kT \frac{f}{T}}$$

$$\overset{\text{Poisson}}{=} \frac{1}{T} \sum_k \hat{x}\left(\frac{f+k}{T} \right)$$

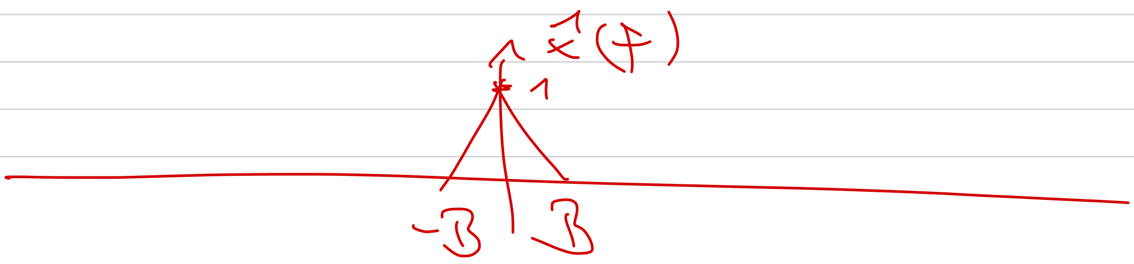
$$\sum_k x(t + kT) = \frac{1}{T} \sum_k \hat{x}(k/T) e^{i2\pi \ell \frac{t}{T}}$$

periodic $\Longleftarrow$ F.S.

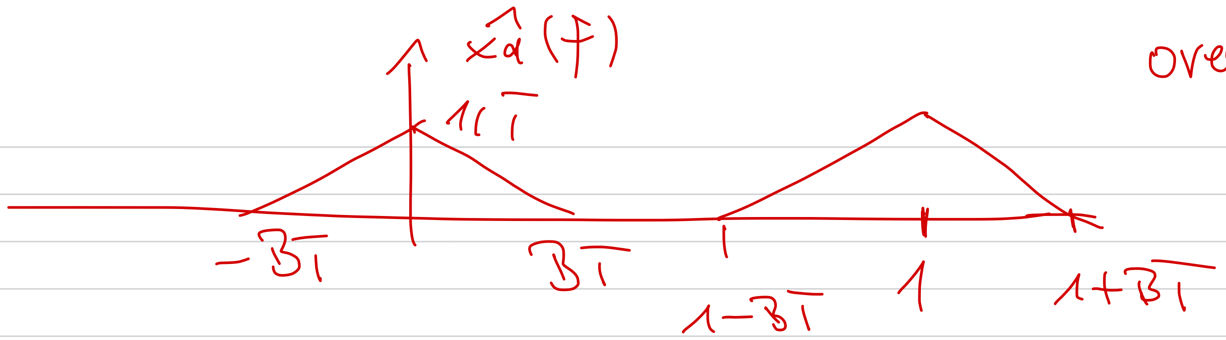
$$t = 0: \quad \sum_k x(kT) = \frac{1}{T} \sum_k \hat{x}(k/T)$$

$$\hat{x}_d(f) = \frac{1}{T} \hat{x}\left(\frac{f}{T}\right)$$

$$\boxed{\hat{x}(f) = T \hat{x}_d(fT)}$$

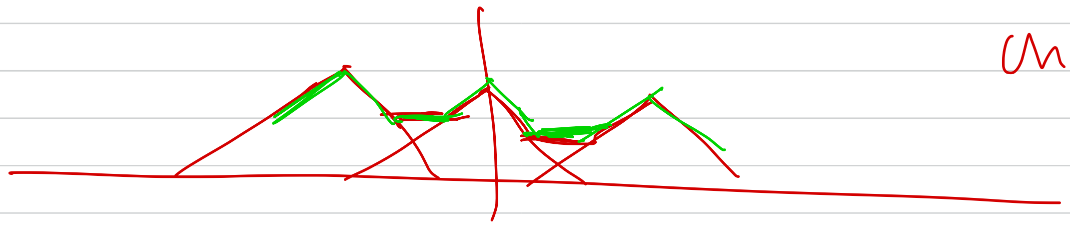


oversampling



critical sampling

undersampling



$$1 - BT \geq BT$$

$$1 \geq 2BT$$

$$\boxed{T \leq \frac{1}{2B}}$$

$$f_s = \frac{1}{T} \geq 2B$$ Nyquist-Shannon sampling theorem

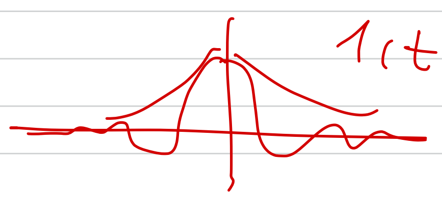
$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\, e^{i2\pi tf}\, df$$

$$= \int_{-\infty}^{\infty} T\, \hat{h}_{LP}(fT) \sum_{k} x(kT)\, e^{-i2\pi kTf}\, e^{i2\pi ft}\, df$$

$$\vdots$$

$$= 2BT \sum_{k} x(kT)\, \overbrace{\mathrm{sinc}(2B(t-kT))}^{g_k(t) = \mathrm{sinc}(2B(t-kT))}.$$

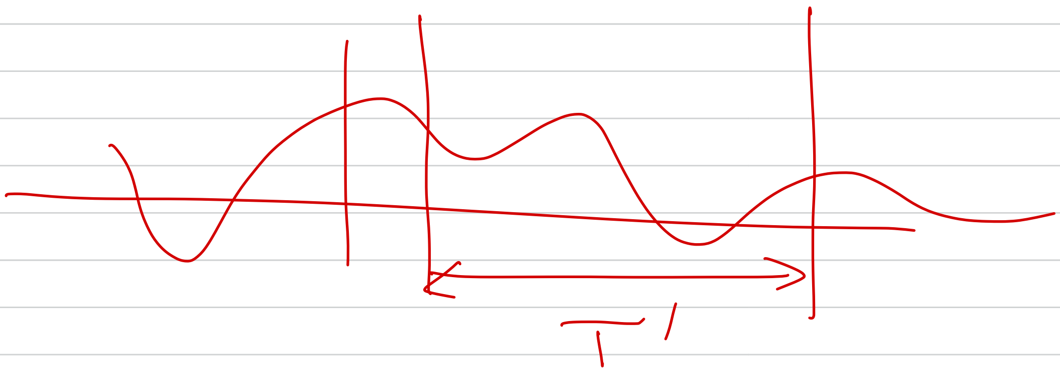
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



$$2B\underset{\substack{\uparrow \\ 1/f_s}}{T} = \frac{2B}{f_s}$$

Landau–Slepian–Pollak

T'



$$\frac{T'}{T} = T'/(1/2B)$$

$$= 2BT'$$

$2BT'$ samples are enough to "uniquely" specify $x(t)$ on this interval

$2B$ samples per second to