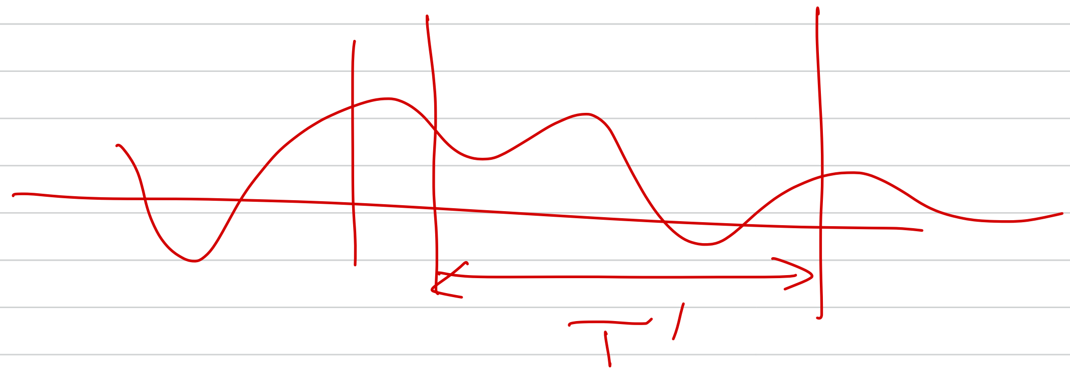
Landau – Slepian – Pollak

T'



$$\frac{T'}{T} = T' / (1/2B) = 2BT'$$

$2BT'$ samples are enough to "uniquely" specify $x(t)$ on this interval

$2B$ samples per second to

Theorem 1.38. Let $x \in \mathcal{L}^2(B)$. Then, $x(t)$ is uniquely specified by its samples $x(kT)$, $k \in \mathbb{Z}$, if $f_s = \frac{1}{T} \geq 2B$. Specifically, we can reconstruct $x(t)$ from $x(kT)$, $k \in \mathbb{Z}$, according to

$$x(t) = 2BT \sum_{k \in \mathbb{Z}} x(kT) \operatorname{sinc}(2B(t - kT)).$$

1.4.1. Sampling theorem as a frame expansion

1. $g_k(t) = 2B \operatorname{sinc}(2B(t - kT))$, $k \in \mathbb{Z}$

2. $$x(kT) = \int_{-B}^{B} \hat{x}(f) e^{i2\pi f t} df \Big|_{t = kT} = \int_{-B}^{B} \hat{x}(f) e^{i2\pi f k T} df$$

$$\overset{\text{F.T. table}}{=} \langle \hat{x}, \hat{g_k} \rangle = \langle x, g_k \rangle$$

$$\hat{g_k}(f) = \begin{cases} e^{-i2\pi k T f}, & |f| \leq B \\ 0, & \text{else} \end{cases}$$

$$x(t) = T \sum_{k \in \mathbb{Z}} \langle x, g_k \rangle g_k(t)$$

$$\|x\|^2 = \langle x, x \rangle = \langle T \sum_{k \in \mathbb{Z}} \langle x, g_k \rangle g_k, x \rangle$$

$$= T \sum_{k \in \mathbb{Z}} |\langle x, g_k \rangle|^2$$

$$\frac{1}{T}\|x\|^2 = \sum_{k\in\mathbb{Z}} |\langle x, g_k\rangle|^2 = \langle Sx, x\rangle$$

$$\langle Sx, x\rangle = A\|x\|^2, \quad A = \frac{1}{T}$$

Conclusion: We have a tight frame expansion regardless of the value of T as long as $\frac{1}{T} \geq 2B$.

$$T: x \to \{\langle x, g_k\rangle\}_{k\in\mathbb{Z}}$$

$$T^*: \{c_k\}_{k\in\mathbb{Z}} \to \sum_k c_k g_k$$

$$S = T^*T \Rightarrow S: x \to \sum_k \langle x, g_k\rangle g_k$$

have established that $S = AI$, $A = 1/T$

$$\Rightarrow \tilde{g}_k(t) = (S^{-1}g_k)(t) = T g_k(t), \; k\in\mathbb{Z}$$

$$\langle g_k, \tilde{g}_k\rangle = T\langle g_k, g_k\rangle = T\|g_k\|^2 \overset{\text{Parseval}}{=} T\|\hat{g}_k\|^2 = 2BT$$

$$\boxed{\langle g_k, \tilde{g}_k\rangle = 2BT} = \frac{2B}{f_s}, \quad f_s = \frac{1}{T}$$

$$\langle g_k, \tilde{g}_k\rangle = 2BT = \frac{2B}{f_s} \overset{\text{critical sampling}}{=} 1 \Rightarrow \{g_k\}_{k\in\mathbb{Z}} \text{ is an exact frame}$$

want to establish that in the case of critical sampling $\{g_k\}_{k\in\mathbb{Z}}$ is, in fact, an ONB.

$$g_k'(t) = \sqrt{T}\, g_k(t)$$

$$x(t) = \sum_{k\in\mathbb{Z}} \langle x, g_k'\rangle g_k'(t) \Rightarrow \text{tight frame with } A = 1$$

$$\|g_k'\|^2 = T\|g_k\|^2 = T\|\hat{g}_k\|^2 = 2BT \overset{\text{critical sampling}}{=} 1$$

$\Rightarrow \{g_k\}_{k\in\mathbb{Z}}$ is an ONB for critical sampling

Chapter 2. Uncertainty relations and sparse signal recovery

$$x(t) \circ\!\!-\!\!\bullet\ \hat{x}(f)$$

$$x(at) \circ\!\!-\!\!\bullet\ \frac{1}{|a|}\hat{x}(f/a)$$

$$\underbrace{\sigma_t}_{\text{time duration}}\ \underbrace{\sigma_f}_{\text{bandwidth}} \geq \text{const.} \quad \Rightarrow \quad \sigma_f \geq \frac{\text{const.}}{\sigma_t}$$

$$x(t) = \int \hat{x}(f)\, e^{i2\pi f t}\, df \qquad f \in \mathbb{R}$$

$$\hat{x}(f) = \int x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi \cdot f} \rangle$$

$$x(t) = \int_{-\infty}^{\infty} x(t')\, \underbrace{\delta(t-t')}_{\text{Dirac } \delta\text{-function}}\, dt' = \langle x(\cdot), \delta(t-\cdot) \rangle$$

$x \in \mathbb{C}^n$, $[F]_{k,l} = \frac{1}{\sqrt{n}} e^{i2\pi \frac{kl}{n}}$, $n \times n$ DFT matrix

$$\hat{x} = Fx = \underbrace{\begin{bmatrix} f_0^H \\ f_1^H \\ \vdots \end{bmatrix}}_{F} x = \begin{bmatrix} \langle x, f_0 \rangle \\ \langle x, f_1 \rangle \\ \vdots \end{bmatrix}$$

$$F^H = [f_0\ f_1 \dots]$$

$$x = Ix = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ & & \ddots \end{bmatrix} x = \begin{bmatrix} \langle x, e_0 \rangle \\ \langle x, e_1 \rangle \\ \vdots \end{bmatrix}$$

Notation: $\mathcal{A} \subseteq \{1, \dots, m\}$

$$\underset{m \times m}{D_{\mathcal{A}}} = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}, \qquad (D_{\mathcal{A}})_{i,i} = \begin{cases} 1, & i \in \mathcal{A} \\ 0, & i \notin \mathcal{A} \end{cases}$$

$\mathcal{U} \in \mathbb{C}^{m \times m}$

$$P_{\mathcal{A}}(\mathcal{U}) = \mathcal{U} D_{\mathcal{A}} \mathcal{U}^H = [u_1 \; u_2 \dots u_m] \begin{pmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{pmatrix} \begin{bmatrix} u_1^H \\ u_2^H \\ \vdots \\ u_m^H \end{bmatrix}$$

$$= \sum_{i \in \mathcal{A}} u_i u_i^H$$

$$W^{\mathcal{U},\mathcal{A}} = R(P_{\mathcal{A}}(\mathcal{U}))$$

$$|||A|||_2 = \max_{x : \|x\|_2 = 1} \|Ax\|_2 \quad \text{, operator 2-norm}$$

$$\|A\|_2 = \sqrt{\mathrm{tr}(AA^H)} \quad \text{, Frobenius norm}$$

$$\Delta_{P,Q}(\mathcal{U}) = |||D_P P_Q(\mathcal{U})|||_2 \overset{\text{Lemma 2.20}}{=} \max_{x \in W^{\mathcal{U},Q} \setminus \{0\}} \frac{\|x_P\|_2}{\|x\|_2}$$

$$x_P = D_P x$$

uncertainty relation: $\Delta_{P,Q}(\mathcal{U}) \leq c < 1$

$$\|x_P\|_2 \leq \underbrace{c}_{<1} \|x\|$$

$$D_P = I \, D_P \, I$$

$$P_Q(\mathcal{U}) = \mathcal{U} D_Q \mathcal{U}^H$$

$$|||P_P(A) P_Q(B)|||_2 = \max_{x : \|x\|_2 = 1} \|P_P(A) P_Q(B) x\|_2$$

$$= \max_{x : \|x\|_2 = 1} \| A D_P A^H B D_Q B^H x\|_2$$

$$= \max_{x : \|x\|_2 = 1} \| D_P \underbrace{A^H B}_{\mathcal{U}} D_Q \underbrace{B^H A}_{\mathcal{U}^H} x\|_2$$

$$= \max_{x:\|x\|_2=1} \| D_P U D_Q U^H x\|_2$$

$$= \max_{x:\|x\|_2=1} \| D_P P_Q(U) x\|_2$$

$$= \||D_P P_Q(U)\||_2 .$$

Lemma 2.21.

$$\frac{\| D_P P_Q(U)\|_2}{\sqrt{\operatorname{rank}(D_P P_Q(U))}} \le \Delta_{P,Q}(U) \le \| D_P P_Q(U)\|_2$$

$$\operatorname{rank}(D_P P_Q(U)) \le \min(|P|, |Q|)$$

$$\operatorname{rank}(D_P U D_Q U^H) \le \text{— " —}$$

$$\left(\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\} \right)$$

$$\Delta_{P,Q}(U) = \sqrt{\operatorname{tr}(D_P P_Q(U))}$$

$$\| D_P P_Q(U)\|_2 = \sqrt{\operatorname{tr}(D_P \underbrace{P_Q(U) P_Q^H(U)}_{P_Q(U)} D_P^H)}$$

$$\boxed{\sqrt{\frac{\operatorname{tr}(D_P P_Q(U))}{\min(|P|,|Q|)}} \le \Delta_{P,Q}(U) \le \sqrt{\operatorname{tr}(D_P P_Q(U))}}$$

Ex. 1. $P = \{1\}$, $Q = \{1, \dots, m\}$, $U = F$

$$= \operatorname{tr}(D_P F D_Q D_Q F^H D_P)$$

$$\sqrt{\operatorname{tr}(D_P P_Q(F))} = \sqrt{\operatorname{tr}(D_P F D_Q F^H)} \qquad F_{k,l} = \frac{1}{\sqrt{m}} e^{i 2\pi \frac{kl}{m}}$$

$$= \sqrt{\sum_{i \in P} \sum_{j \in Q} |F_{ij}|^2} = \sqrt{\frac{|P||Q|}{m}}$$

$$\boxed{\sqrt{\frac{\max(|P|,|Q|)}{m}} \leq \Delta_{P,Q}(\mathcal{F}) \leq \sqrt{\frac{|P||Q|}{m}}}$$

$$1 = \sqrt{\frac{m}{m}} \leq \Delta_{P,Q}(\mathcal{F}) \leq \sqrt{\frac{1\cdot m}{m}} = 1$$

$$\boxed{\Delta_{P,Q}(\mathcal{F}) = 1}$$

Ex. 2. saturating the lower bound, take n to divide m

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \right\}$$

$$Q = \{\ell+1, \ldots, \ell+n\}, \text{ interpreted circularly}$$

$$\left(\sum_k \delta(t-kT) \multimap \frac{1}{T} \sum_k \delta(f - k/T) \right)$$

upper bound: $\sqrt{\frac{|P||Q|}{m}} = \frac{n}{\sqrt{m}}$

lower bound: $\sqrt{\frac{n}{m}}$

$$\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(\mathcal{F}) \leq \frac{n}{\sqrt{m}}$$

Lemma 2.1. Let n divide m and consider

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \ldots, m \right\}$$

$$Q = \{\ell+1, \ldots, \ell+n\}$$

with $\ell \in \{1, \ldots, m\}$ and Q interpreted circularly in $\{1, \ldots, m\}$.
Then $\Delta_{P,Q}(\mathcal{F}) = \sqrt{n/m}$.

Proof.

$$\Delta_{P,Q}(\mathcal{F}) = |||D_P P_Q(\mathcal{F})|||_2 \quad \leftarrow \text{Lemma 2.20.}$$

$$= |||P_Q(\mathcal{F}) D_P|||_2$$

$$= \max_{x : \|x\|_2 = 1} \| \cancel{F} D_Q F^H D_P x \|_2$$

$$= \max_{x : x \neq 0} \frac{\| D_Q F^H D_P x \|_2}{\|x\|_2}$$

$$= \max_{\substack{x : x = x_P \\ x \neq 0}} \frac{\| D_Q F^H x \|_2}{\|x\|_2}$$

$$\| D_Q F^H x \|_2 = \frac{1}{m} \sum_{q \in Q} \left| \sum_{p \in P} x_p e^{\frac{2\pi i p q}{m}} \right|^2$$

$$= \frac{1}{m} \sum_{q \in Q} \left| \sum_{s=1}^{n} x_{msin} e^{\frac{2\pi i s \cancel{m} q}{\cancel{m} n}} \right|^2$$

$$= \frac{1}{m} \sum_{q \in Q} \left| \sum_{s=1}^{n} y_s e^{\frac{2\pi i s q}{n}} \right|^2$$

$$= \frac{n}{m} \| F^H y \|_2 = \frac{n}{m} \|y\|_2$$

$$\|y\|_2 = \|x\|_2$$

$$\Delta^2_{P,Q}(F) = \frac{n}{m}$$

$$\Rightarrow \Delta_{P,Q}(F) = \sqrt{\frac{n}{m}} . \quad \text{q.e.d.}$$