$$= \max_{x:\|x\|_2=1} \|\cancel{F} D_Q F^H D_P x\|_2$$

$$= \max_{x:x\neq 0} \frac{\|D_Q F^H D_P x\|_2}{\|x\|_2}$$

$$= \max_{\substack{x:x=x_P\\ x\neq 0}} \frac{\|D_Q F^H x\|_2}{\|x\|_2}$$

$$\|D_Q F^H x\|_2 = \frac{1}{m}\sum_{q\in Q}\Big|\sum_{p\in P} x_p e^{\frac{2\pi i p q}{m}}\Big|^2$$

$$= \frac{1}{m}\sum_{q\in Q}\Big|\sum_{s=1}^{n} x_{msin} e^{\frac{2\pi i s \cancel{m} q}{\cancel{m} n}}\Big|^2$$

$$= \frac{1}{m}\sum_{q\in Q}\Big|\sum_{s=1}^{n} y_s e^{\frac{2\pi i s q}{n}}\Big|^2$$

$$= \frac{n}{m}\|F^H y\|_2 = \frac{n}{m}\|y\|_2$$

$$\|y\|_2 = \|x\|_2$$

$$\Delta_{P,Q}^2(F) = \frac{n}{m}$$

$$\Rightarrow \Delta_{P,Q}(F) = \sqrt{\frac{n}{m}}.\quad \text{q.e.d.}$$

2.2.2. Coherence-based uncertainty relation

Def. 2.3. For $A=(a_1 \ldots a_n)\in\mathbb{C}^{m\times n}$ with columns $\|\cdot\|_2$-normalized to 1, the coherence is defined as

$$\mu(A) = \max_{i\neq j} |a_i^H a_j|.$$

Lemma 2.4. Let $U\in\mathbb{C}^{m\times m}$ and $P,Q\subseteq[1,\ldots,m]$. Then,

$$\Delta_{P,Q}(U) \leq \sqrt{|P||Q|}\,\mu([I_m\ U]).$$

Proof

$$\Delta_{P,Q}^2(U) \leq \operatorname{tr}(D_P P_Q(U))$$

$$= \operatorname{tr}(D_P U D_Q U^H)$$

$$= \operatorname{tr}(D_P U D_Q | D_Q U^H D_P)$$

$$= \sum_{i \in P} \sum_{l \in Q} |U_{i,l}|^2$$

$$\leq |P||Q| \max_{k,l} |U_{k,l}|^2$$

$$= |P||Q| \, \mu^2([I \; U]) . \quad \square$$

$$\mu([I \; F]) = \frac{1}{\sqrt{m}}$$

$$\Delta_{P,Q}(F) \leq \sqrt{|P||Q|} \, \frac{1}{\sqrt{m}}$$

2.2.3. Concentration inequalities

We want to characterize the "concentration" of p and q in

$$p = Fq \; (= x)$$

$$\hat{q} = Fq$$

$$Ip = Fq \; (= x)$$

$$Ap = Bq \; (= x)$$

$$\mu([A \; B])$$

Def. 2.5. Let $P \subseteq \{1, \ldots, m\}$ and $\varepsilon_P \in [0,1]$. The vector $x \in \mathbb{C}^m$ is said to be ε_P-concentrated if

$$\|x - x_P\|_2 \leq \varepsilon_P \|x\|_2 .$$

$$(x_P = D_P x)$$

$$\frac{\|x - x_P\|_2}{\|x\|_2} \leq \varepsilon_P .$$

$Ap = Bq \quad (=x)$

$p = \underbrace{A^H B}_{U} q \, , \quad p = Uq$

Lemma 2.6. Let $U \in \mathbb{C}^{m \times m}$ be unitary and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exist a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $p = Uq$. Then,

$$\Delta_{P,Q}(U) \geq [1 - \varepsilon_P - \varepsilon_Q]_+ .$$

Proof.

$$\|p - P_Q(U) p_P\|_2 \leq \|p - P_Q(U) p\|_2 + \|P_Q(U) p_P - P_Q(U) p\|_2$$

$$\leq \|p - P_Q(u) p\|_2 + |||P_Q(U)||| \, \|p_P - p\|_2$$

use $p = uq$ $\quad \leq \|Uq - P_Q(u) Uq\|_2 + \|p_P - p\|_2$

$$= \|Uq - U \underbrace{D_Q \underbrace{U^H U}_{I} q}_{q_Q}\|_2 + \|p_P - p\|_2$$

$$= \|q - q_Q\|_2 + \|p - p_P\|_2$$

$$\leq \varepsilon_q \|q\|_2 + \varepsilon_P \|p\|_2 \quad (p = uq)$$

$$= \varepsilon_q \|q\|_2 + \varepsilon_P \|q\|_2$$

$$= (\varepsilon_P + \varepsilon_q) \|p\|_2 .$$

$$\|P_Q(U) p_P\|_2 \overset{\pm p,\ \text{rev.}\ \Delta\text{-ineq.}}{\geq} [\|p\|_2 - \|p - P_Q(U) p_P\|_2]_+$$

$$\geq \|p\|_2 [1 - \varepsilon_P - \varepsilon_Q]_+$$

$$\|P_Q(U) D_P \frac{p}{\|p\|_2}\| \geq [1 - \varepsilon_P - \varepsilon_Q]_+ .$$

$$\Delta_{P,Q}(U) \geq [1 - \varepsilon_P - \varepsilon_Q]_+ . \quad \square$$

Corollary 2.7. Let $A, B \in \mathbb{C}^{m\times m}$ be unitary and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exist a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $Ap = Bq$. Then,

$$|P|\,|Q| \geq \frac{[1-\varepsilon_P-\varepsilon_Q]_+^2}{\mu^2([A\ B])}.$$

Proof.

$$[1-\varepsilon_P-\varepsilon_Q]_+ \leq \Delta_{P,Q}(U) \leq \sqrt{|P||Q|}\,\mu([I\ U]),$$

where $U = A^H B$.

$$\mu([I\ A^H B]) = \mu([A\ B])$$

$$|P||Q| \geq \frac{[1-\varepsilon_P-\varepsilon_Q]_+^2}{\mu^2([A\ B])}. \qquad \square$$

$Ap = Bq$

$p = Fq$

Special case: $\varepsilon_P = \varepsilon_Q = 0$

Corollary 2.8. Let $A, B \in \mathbb{C}^{m\times m}$ be unitary. If $Ap = Bq$ for nonzero $p, q \in \mathbb{C}^m$, then

$$\|p\|_0 \|q\|_0 \geq \frac{1}{\mu^2([A\ B])}.$$

$A = I, B = F \Rightarrow \|p\|_0 \|q\|_0 \geq m. \leftarrow$ Donoho & Stark, 1991.

Elad–Bruckstein uncertainty relation.

2.2.4. Noisy recovery in $(\mathbb{C}^m, \|\cdot\|_2)$

$p, n \in \mathbb{C}^m$

lose all the samples entries indexed by $\mathcal{P}$

observe $y = p_{\mathcal{P}^c} + n$, $\quad \mathcal{P}^c = \{1, \dots, m\} \setminus \mathcal{P}$

additive noise

Lemma 2.9. Let $U \in \mathbb{C}^{m \times m}$ be unitary, $\mathcal{Q} \subseteq \{1, \dots, m\}$, $p \in \mathcal{W}^{U, \mathcal{Q}}$, and consider

$$y = p_{\mathcal{P}^c} + n,$$

where $n \in \mathbb{C}^m$ and $\mathcal{P}^c = \{1, \dots, m\} \setminus \mathcal{P}$. If $\Delta_{\mathcal{P}, \mathcal{Q}}(U) < 1$, then there exists a matrix $L \in \mathbb{C}^{m \times m}$ s.t.

$$\|Ly - p\|_2 \leq C \|n_{\mathcal{P}^c}\|_2$$

with $C = 1 / (1 - \Delta_{\mathcal{P}, \mathcal{Q}}(U))$. In particular,

$$|\mathcal{P}||\mathcal{Q}| < \frac{1}{\mu^2([I \; U])}$$

guarantees $\Delta_{\mathcal{P}, \mathcal{Q}}(U) < 1$.

$$\left(\Delta_{\mathcal{P}, \mathcal{Q}}(U) = \max_{x \in \mathcal{W}^{U, \mathcal{Q}} \setminus \{0\}} \frac{\|x_{\mathcal{P}}\|_2}{\|x\|_2} . \right)$$

$$\frac{\|x_{\mathcal{P}}\|_2}{\|x\|_2} < \Delta_{\mathcal{P}, \mathcal{Q}}(U)$$

Proof. For $\Delta_{\mathcal{P}, \mathcal{Q}}(U) < 1$, $(I - D_{\mathcal{P}} P_{\mathcal{Q}}(U))$ is invertible with

$$|||(I - D_{\mathcal{P}} P_{\mathcal{Q}}(U))^{-1}|||_2 \leq \frac{1}{1 - |||D_{\mathcal{P}} P_{\mathcal{Q}}(U)|||_2} = \frac{1}{1 - \Delta_{\mathcal{P}, \mathcal{Q}}(U)} .$$

$$\left((I - A)^{-1} = \sum_{k=0}^{\infty} A^k , \qquad |||A||| < 1 \right.$$

$$|||(I-A)^{-1}||| \le \sum_{k=0}^{\infty} |||A^k||| = \sum_{k=0}^{\infty} |||A|||^k = \frac{1}{1-|||A|||} \Big)$$

$$L = (I - D_p P_\Theta(u))^{-1} D_{p^c}$$

$$\underline{\underline{L p_{pc}}} = (I - D_p P_\Theta(u))^{-1} \underbrace{D_{pc}\, p_{pc}}_{p_{pc}}$$

$$= (I - D_p P_\Theta(u))^{-1} (\underbrace{I - D_p}_{D_{pc}}) p$$

$$= (I - D_p P_\Theta(u))^{-1} (I - D_p P_\Theta(u)) p$$

$$\underline{\underline{= p}}$$

$$\|Ly - p\|_2 = \| L(p_{pc} + n) - p\|_2$$

$$= \| \underbrace{L p_{pc}}_{p} + Ln - p\|_2 = \|Ln\|_2$$

$$= \|(I - D_p P_\Theta(u))^{-1} \underbrace{D_{pc} n}_{n_{pc}}\|_2$$

$$\le \underbrace{|||(I - D_p P_\Theta(u))^{-1}|||_2}\, \|n_{pc}\|_2$$

$$\le \underbrace{\frac{1}{1 - \Delta_{P,\Theta}(u)}}_{C} \|n_{pc}\|_2 . \quad \square$$

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \dots, \frac{(n-1)m}{n}, m \right\}$$ <u>we lose n samples</u>

n divides m

$$Q = \{ l+1, \dots, l+n \}$$ <u>n-sparse</u>

$\mathcal{U} = \mathcal{F}$

$\Delta_{PIO}(\mathcal{F}) = \sqrt{n/m}$

$\underbrace{n \le m/2} \Rightarrow \Delta_{PIO}(\mathcal{F}) < 1$

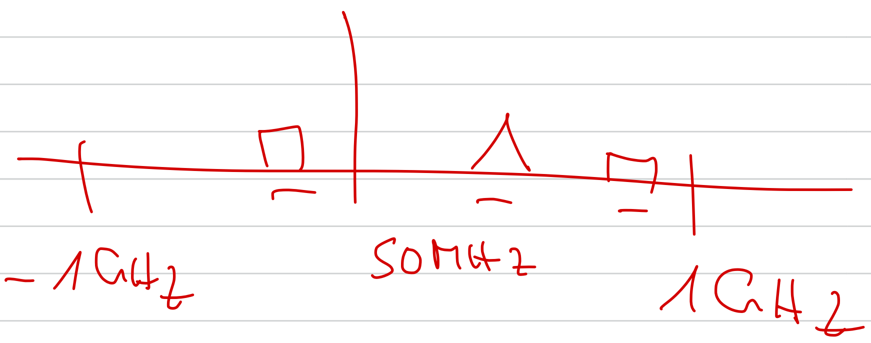
We can lose "half of the ambient dimension." "linear threshold"

upper-bound: $\Delta_{PIO}(\mathcal{F}) \le \frac{n}{\sqrt{m}} < 1$

$n < \sqrt{m}$

$n^2 < m$ "square-root bottleneck"

Chapter 3. Compressed Sensing



Discrete Fourier transform

$\theta \to \theta + 1$

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi\theta n}, \quad \theta \in [0,1), \quad \hat{x}(\theta+1) = \hat{x}(\theta)$$

$$\left(\begin{array}{l} \hat{x}(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}\, dt \\ x(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{i2\pi f t}\, df \end{array} \right), \quad \hat{x}(\theta) = \sum_{n=0}^{N-1} x[n] e^{-i2\pi\theta n}$$

$$\hat{x}\left(\frac{k}{N}\right)$$

$$\hat{x}[k] := \frac{1}{\sqrt{N}} \hat{x}\left(\frac{k}{N}\right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \underbrace{e^{-i2\pi \frac{k}{N} n}}_{\omega_N^{kn}}, \quad \omega_N = e^{-i2\pi/N}.$$