$U = F$

$\Delta_{P,\Theta}(F) = \sqrt{n/m}$

$\underbrace{n \le m/2} \Rightarrow \Delta_{P,\Theta}(F) < 1$

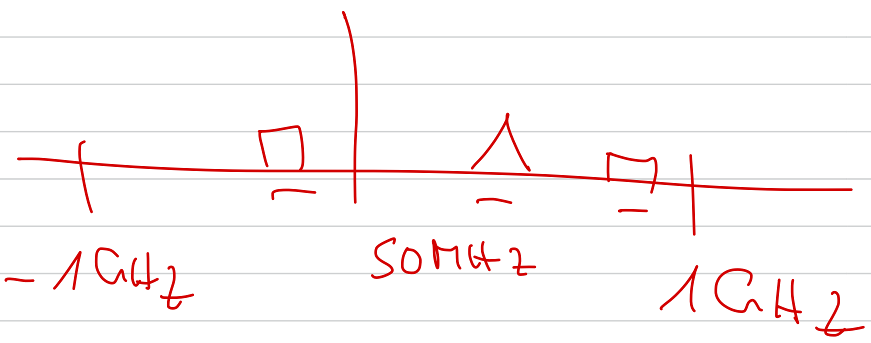
We can lose "half of the ambient dimension." "linear threshold"

upper-bound : $\Delta_{P,\Theta}(F) \le \frac{n}{\sqrt{m}} < 1$

$n < \sqrt{m}$

$n^2 < m$ "square-root bottleneck"

Chapter 3. Compressed Sensing



Discrete Fourier transform

$\theta \to \theta + 1$

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi\theta n}, \quad \theta \in [0,1), \quad \hat{x}(\theta+1) = \hat{x}(\theta)$$

$$\left(\begin{array}{l} \hat{x}(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \, dt \\ x(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{i2\pi f t} \, df \end{array} \right), \quad \hat{x}(\theta) = \sum_{n=0}^{N-1} x[n] e^{-i2\pi\theta n}$$

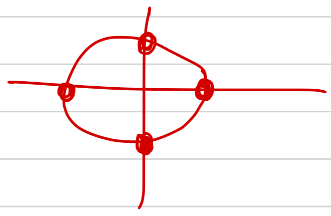
$$\hat{x}\left(\frac{k}{N}\right), \quad k = 0, 1, \dots, N-1$$

$$\hat{x}[k] := \frac{1}{\sqrt{N}} \hat{x}\left(\frac{k}{N}\right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \underbrace{e^{-i2\pi \frac{k}{N} n}}_{\omega_N^{kn}}, \quad \omega_N = e^{-i2\pi/N}.$$

$$\hat{x} = \frac{1}{\sqrt{N}} \begin{pmatrix} \hat{x}(\frac{0}{N}) \\ \hat{x}(1/N) \\ \vdots \\ \hat{x}(\frac{N-1}{N}) \end{pmatrix} = \frac{1}{\sqrt{N}} \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N & & \vdots \\ \vdots & \omega_N^2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & & \end{pmatrix}}_{:= F_N} \underbrace{\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}}_{x} \qquad \square$$

$\hat{x} = F_N x$, e.g. $\langle \cdot, \cdot \rangle$ between first 2 columns of F_N

$$\frac{1}{N} \sum_{l=0}^{N-1} 1 \cdot e^{-i2\pi \frac{l}{N}} =$$



F_N is a unitary matrix , $F_N F_N^H = F_N^H F_N = I_N$

$\hat{x} = F_N x \quad | F_N^H \cdot$

$F_N^H \hat{x} = F_N^H F_N x = x \Rightarrow$

$\hat{x} = F_N x$ forward

$x = F_N^H \hat{x}$ inverse

$$\boxed{\begin{aligned} x[n] &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}[k]\, \omega_N^{-kn} \\ \hat{x}[k] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, \omega_N^{kn} \end{aligned}}$$

Oversampling

$\frac{1}{\sqrt{M}} \hat{x}(k/M)$, $k = 0, 1, \dots, M-1$, $M > N$

$$\frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} x[n] \underbrace{e^{-i2\pi \frac{k}{M} n}}_{\omega_M^{kn}} = \frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} x[n]\, \omega_M^{kn}$$

$$\frac{1}{\sqrt{M}} \begin{pmatrix} \hat{x}(0/M) \\ \hat{x}(1/M) \\ \vdots \\ \hat{x}(\frac{M-1}{M}) \end{pmatrix} = \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_M & & \\ \vdots & \omega_M^2 & & \\ \vdots & \vdots & & \\ 1 & \omega_M^{M-1} & & \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

$\underbrace{\qquad}_{\hat{x}}$ $\underbrace{\qquad\qquad}_{M \times N}$ $\underbrace{\qquad}_{x}$

$F_0 \in \mathbb{C}^{M \times N}$, contains the first N columns of the $M \times M$ DFT matrix F_M

F_0 has orthonormal columns by virtue of being given by the first N columns of F_M (recall that $M > N$)

Moore-Penrose pseudo-inverse $F_0^{\#} = (F_0^H F_0)^{-1} F_0^H$

$$F_0^{\#} \hat{x} = \underbrace{(F_0^H F_0)^{-1} F_0^H F_0}_{I_N} x = x$$

$$x = \underbrace{(F_0^H F_0)^{-1}}_{I} F_0^H \hat{x} = F_0^H \hat{x}$$

$M \times N$

$$\boxed{\begin{aligned} \hat{x} &= F_0 x \\ x &= F_0^H \hat{x} \end{aligned}}$$

$\hat{x} = \underbrace{F_0}_{T} x$, $\hat{x} = T x$, $S = T^H T = I \Rightarrow$ tight frame

T: analysis operator

S: frame operator

Undersampling

$$\frac{1}{\sqrt{M}} \hat{x}(k/M), \quad k = 0, 1, \dots, M-1, \quad M < N$$

$$\frac{1}{\sqrt{M}} \begin{pmatrix} \hat{x}(0/M) \\ \hat{x}(1/M) \\ \vdots \end{pmatrix} = \underbrace{\frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega_M & & \\ \vdots & \omega_M^2 & & \\ 1 & \vdots\ \omega_M^{M-1} & & \end{pmatrix}}_{M \times N} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

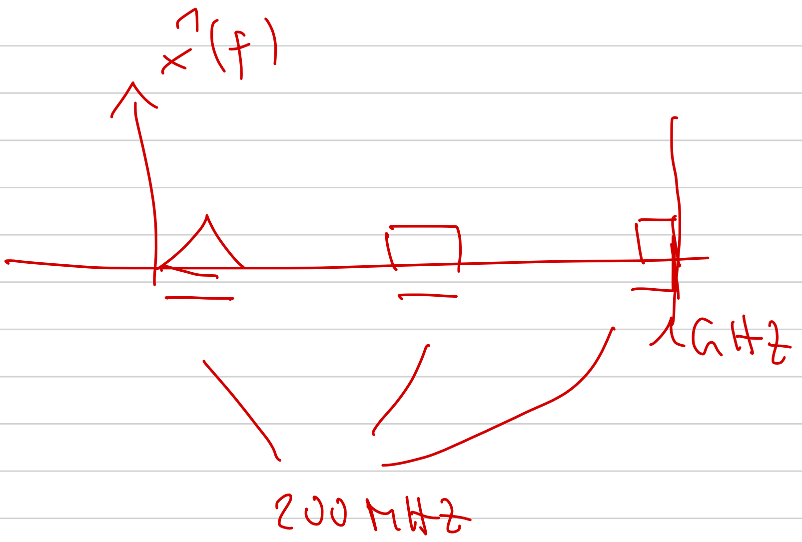
$$\hat{x} = F_n x$$

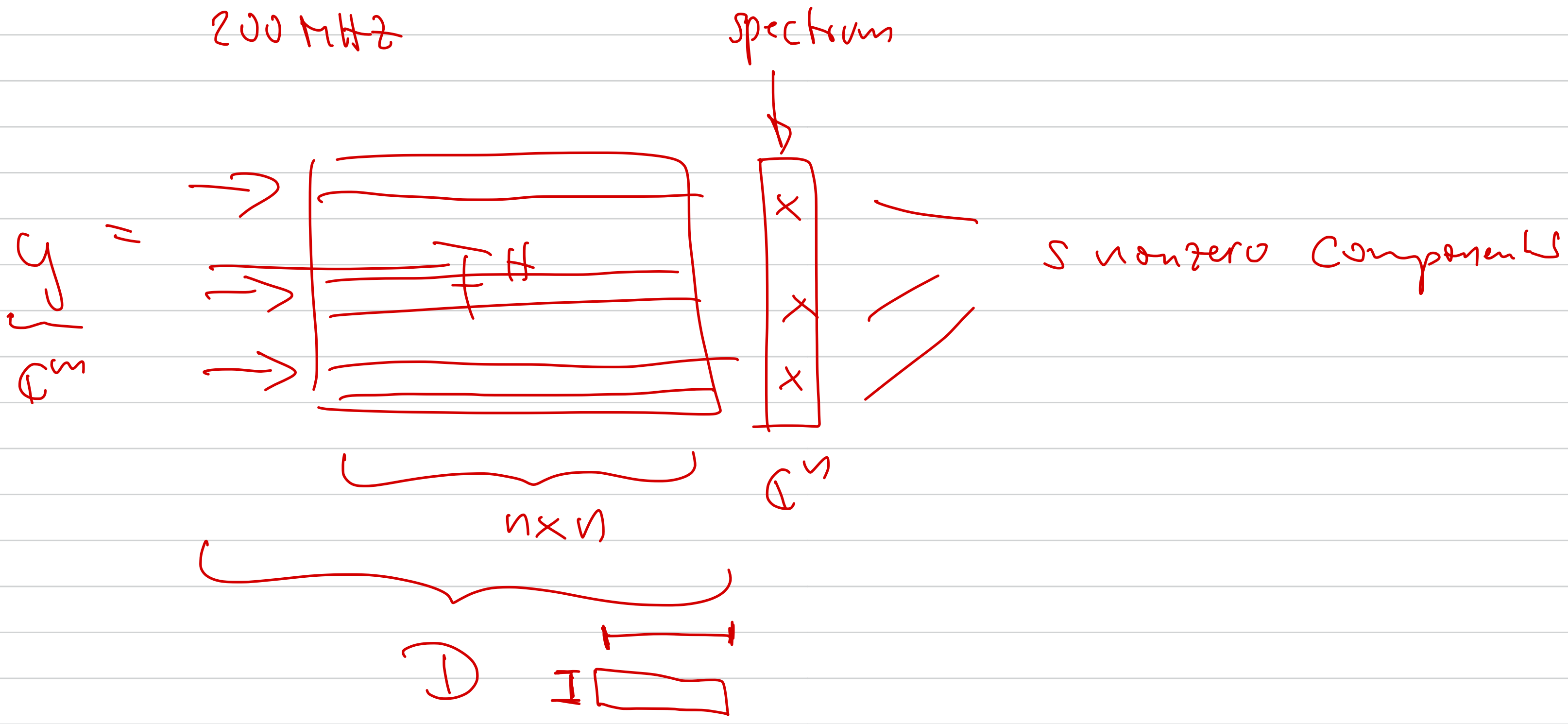
3.2. Compressed sensing

$$\underset{\in \mathbb{C}^m}{y} = \underset{m \times n}{D}\, \underset{\mathbb{C}^n}{x}$$

$m < n$ corresponds to the undersampled case

structure: sparsity, we consider vectors that have few nonzero entries





known support set: locations of nonzero entries of x are known

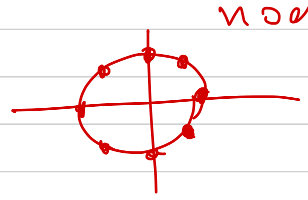
unknown — " — unknown

1. known support set: we can apply the following universal sampling pattern, just select the first $m = s$ rows of F^H, the resulting D-matrix is a Vandermonde matrix

$$\underset{L \times L}{V} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_L \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & \cdots & z_L^{L-1} \end{pmatrix}, \quad z_1, \ldots, z_L \text{ "nodes" of } V$$

$$F = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \text{—} & 1 \\ 1 & e^{-i2\pi/N} & - & \\ 1 & e^{-i2\pi 2/N} & & \\ 1 & \vdots & & \end{pmatrix}$$

nodes



$$\left| \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \right| = 1 \cdot x_2 - 1 \cdot x_1 = x_2 - x_1 \qquad |V| = \prod_{1 \le i < j \le n} (x_j - x_i)$$

$$\left| \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \right| = (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$$

2. unknown support set

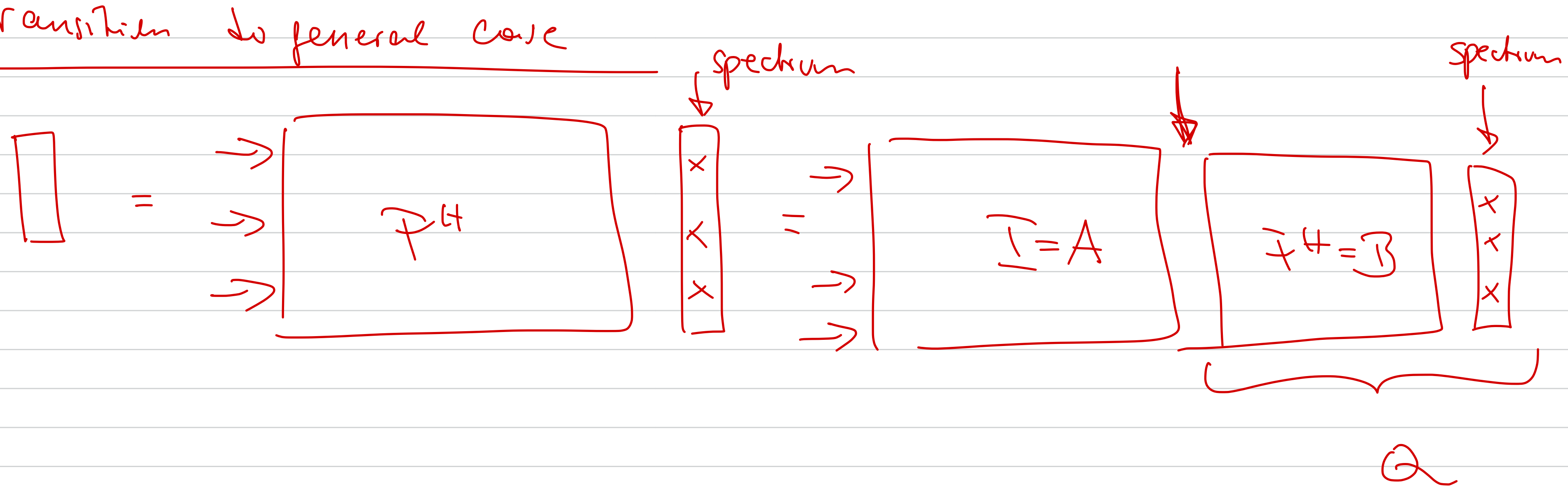
$y_1 = D_1 x_1$

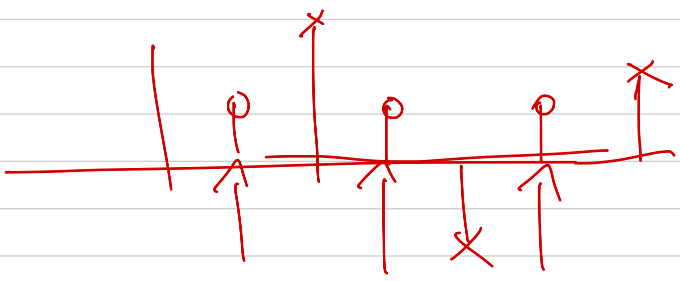
$y_2 = D_2 x_2$

$$y_1 - y_2 = Dx_1 - Dx_2 = D(\underbrace{\overbrace{x_1 - x_2}^{\neq 0}}_{(2s)\text{-sparse}}) \neq 0$$

Take the first $2s$ rows, by virtue of F being Vandermonde, the resulting D-matrix will be a Vandermonde matrix $\Rightarrow$ of full rank $\Rightarrow$ uniqueness of recovery, in principle!

Transition to general case





sampling set

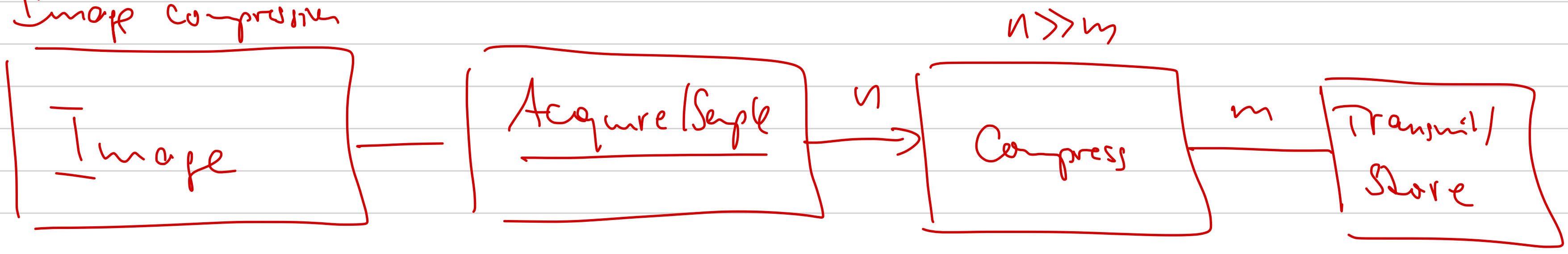
$Ap = Bq = x$

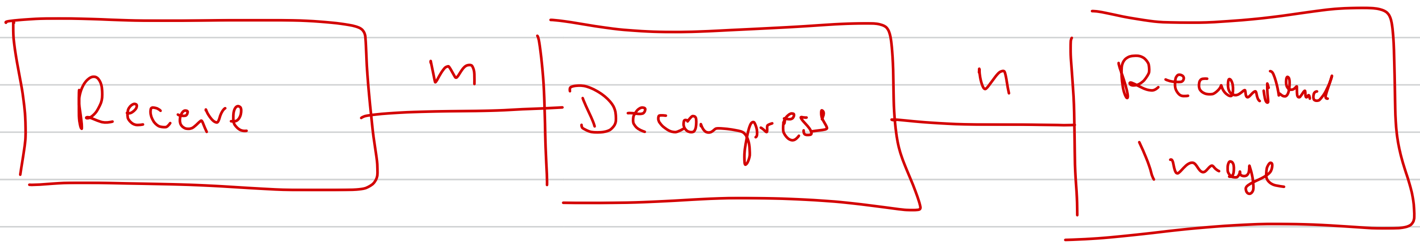
x is s-sparse in B

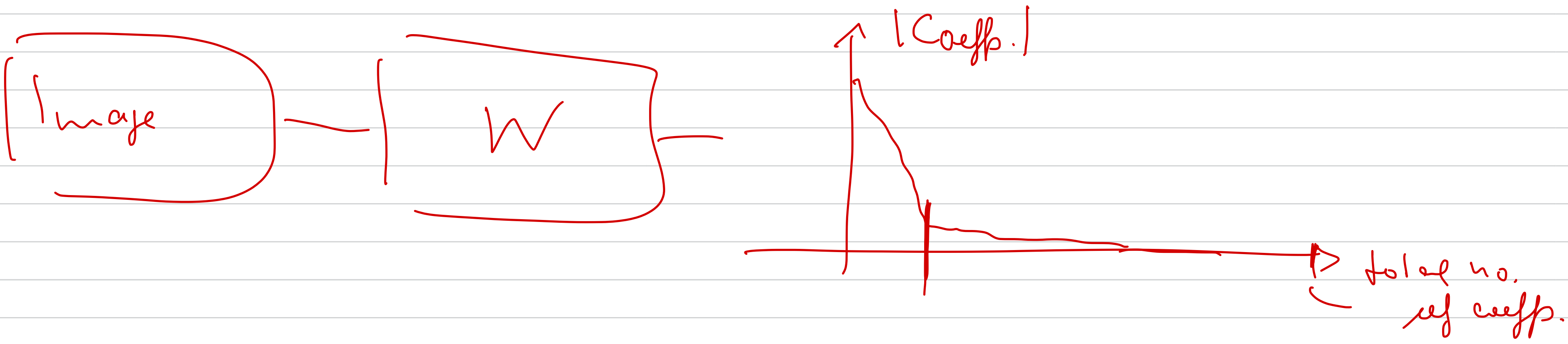
$p = A^H \underbrace{Bq}_{u}$

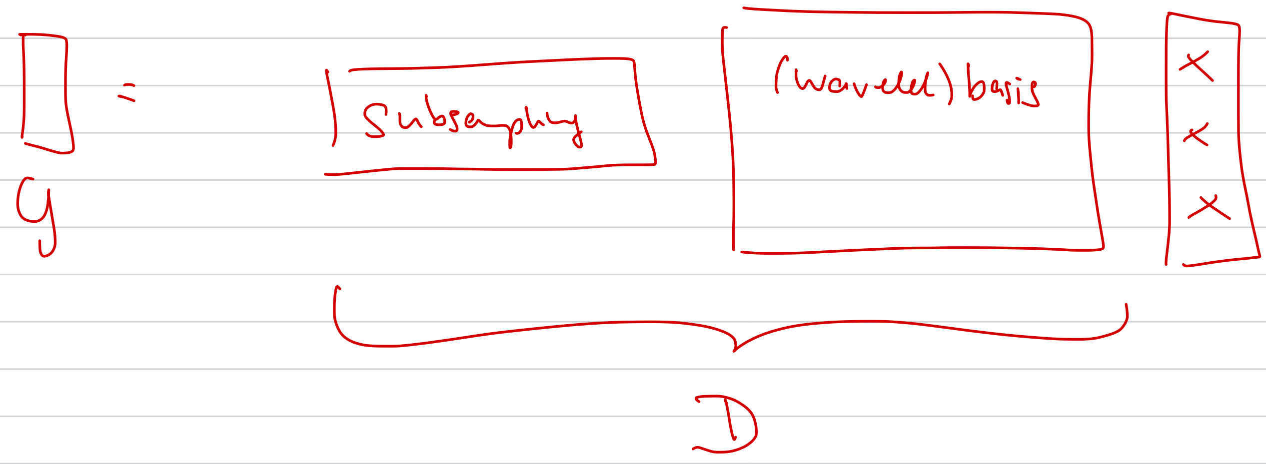
$$\Delta_{p,q}(H) = \max_{x \in W_{H,0}} \frac{\|x\|_p}{\|x\|_q}$$

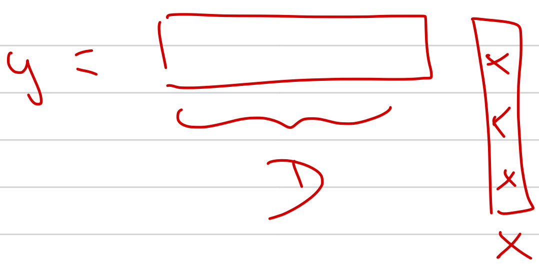
Image compression





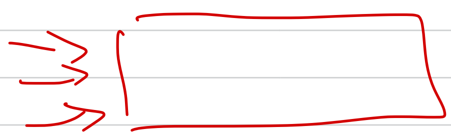






$x_1, x_2 \ldots$ s-sparse

$$y_1 - y_2 = Dx_1 - Dx_2 = D\underbrace{(x_1 - x_2)}_{(2s)\text{-sparse}} \neq 0$$



Def. 3.1. The spark of a matrix A denoted by spark(A) is defined as the cardinality of the smallest set of linearly

dependent columns.

We can uniquely recover s-sparse x from $y = Dx$ if

$$s < \frac{\operatorname{spark}(D)}{2}$$

3.3. The recovery problem P0

$y = Dx \in \mathbb{C}^m$, D: $m \times n$

$\hat{x}$ ... consistent

$\hat{x} \in \{ \{x\} + \mathcal{N}(D) \}$

$\operatorname{spark}(D) > 2s$

$Dx' = 0 \Rightarrow \|x'\|_0 > 2s$

$\hat{x} = x + x'$, $x' \in \mathcal{N}(D)$

$\hat{x}$: consistent; x: correct sol. $\|x\|_0 \le s$

(P0):

find $\operatorname{argmin} \|\hat{x}\|_0$ subject to $y = D\hat{x}$

$m \geq s$ ... reference

Theorem 3.2. (P0) applied to $y = Dx$ recovers x if

$$\|x\|_0 \le s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right).$$

$$\mu(D) = \max_{r \neq l} |\langle d_r, d_l \rangle|$$

Proof. We will show that $\operatorname{spark}(D) > 1 + 1/\mu(D)$. Consider $x \in \mathbb{C}^n$ with $\|x\|_0 = \operatorname{spark}(D)$ and $Dx = 0$.

$$Dx = 0 \iff d_l x_l = - \sum_{r \neq l} d_r x_r \quad | \, d_l^{\#} \cdot$$

$$\sum_{l=1}^{n} d_l x_l = 0$$

$$\underbrace{d_l^{\#} d_l}_{=1} x_l = - \sum_{r \neq l} \langle d_r, d_l \rangle x_r$$

$$|x_l| = \left| \sum_{r \neq l} \langle d_r, d_l \rangle x_r \right|$$

$$\le \sum_{r \neq l} |\langle d_r, d_l \rangle| \, |x_r|$$