We can uniquely recover s-sparse x from $y = Dx$ if

$$s < \frac{\operatorname{spark}(D)}{2}$$

3.3. The recovery problem P0

$$y = \underset{m \times n}{D} \, x \in \mathbb{C}^n$$

$\hat{x}$ ... consistent

$\hat{x} \in \{ \{x\} + \mathcal{N}(D) \}$

$\operatorname{spark}(D) > 2s$

$Dx' = 0 \Rightarrow \|x'\|_0 > 2s$

$$\underset{\text{consistent}}{\hat{x}} = \underset{\substack{\text{correct sol.} \\ \|x\|_0 \le s}}{x} + \underset{\in \mathcal{N}(D)}{x'}$$

(P0):

find $\operatorname{argmin} \|\hat{x}\|_0$ subject to $y = D\hat{x}$

$m \geq s$ ... reference

Theorem 3.2. (P0) applied to $y = Dx$ recovers x if

$$\|x\|_0 \le s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right).$$

$$\mu(D) = \max_{r \neq \ell} |\langle d_r, d_\ell \rangle|$$

Proof. We will show that $\operatorname{spark}(D) > 1 + 1/\mu(D)$. Consider $x \in \mathbb{C}^n$ with $\|x\|_0 = \operatorname{spark}(D)$ and $Dx = 0$.

$$Dx = 0 \Leftrightarrow d_\ell x_\ell = - \sum_{r \neq \ell} d_r x_r \quad | \, d_\ell^{\#} .$$

$$\sum_{\ell'=1}^{n} d_{\ell'} x_{\ell'} = 0$$

$$\underbrace{d_\ell^{\#} d_\ell}_{=1} x_\ell = - \sum_{r \neq \ell} \langle d_r, d_\ell \rangle x_r$$

$$|x_\ell| = \left| \sum_{r \neq \ell} \langle d_r, d_\ell \rangle x_r \right|$$

$$\le \sum_{r \neq \ell} |\langle d_r, d_\ell \rangle| \, |x_r|$$

$$\le \mu(D) \sum_{r\neq \ell} |x_r|$$

$+\ \mu(D)|x_\ell|$ on both sides

$$(1+\mu(D))\,|x_\ell| = \mu(D)\,\|x\|_1$$

$\sum$ over all ℓ for which $x_\ell \neq 0$

$$(1+\mu(D)) \underbrace{\sum_{\ell : x_\ell \neq 0} |x_\ell|}_{\cancel{\|x\|_1}} = \mu(D)\,\mathrm{spark}(D)\,\cancel{\|x\|_1}$$

$$\mathrm{spark}(D) > \left(1 + \frac{1}{\mu(D)}\right).\quad \text{q.e.d.}$$

3.4. Basis Pursuit (BP)

(P1) Find $\arg\min \|\hat{x}\|_1$ subject to $y = D\hat{x}$

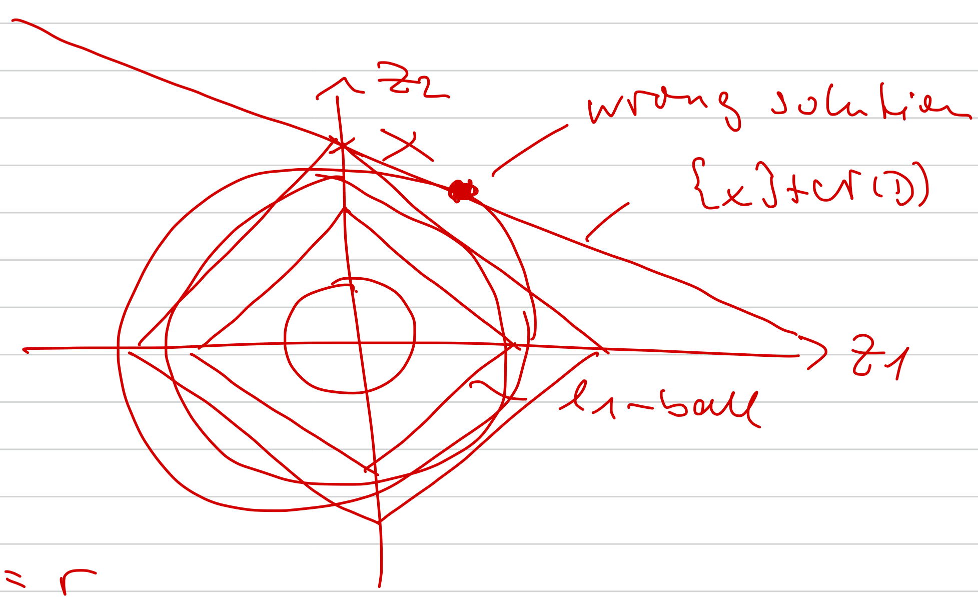
Early results on ℓ_1-reconstruction

- Logan, 1965
- Donoho & Logan, 1992

$$\arg\min \|\hat{x}\|_1 \text{ subject to } y = D\hat{x}$$

$$\Updownarrow$$

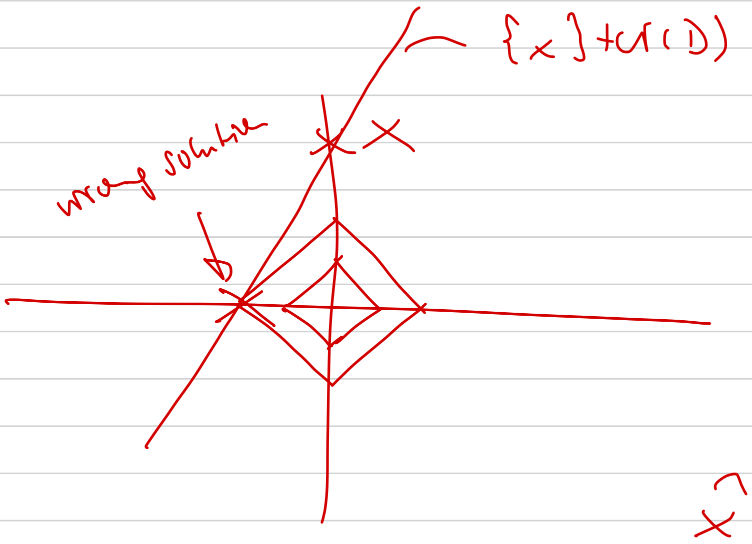
$$\arg\min \|\hat{x}\|_1 \text{ subject to } \hat{x} \in \{\{x\} + \mathcal{N}(D)\}$$



ℓ_1-ball: $|z_1| + |z_2| = \text{const.}$

$|z_2| = \text{const.} - |z_1|$

Of course, this can also go wrong



$\hat{x} \in \{\{x\}+\mathcal{N}(D)\}$

Def. 3.3. We denote

$$P_1(S,D) := \max_{x\in\mathcal{N}(D),\, x\neq 0} \frac{\sum_{k\in S}|x_k|}{\sum_k |x_k|}.$$

Theorem. Arbitrarily fix x with support set S and let $y=Dx$. If $P_1(S,D) < 1/2$, then x is the unique solution to

(P1) find any min $\|\hat{x}\|_1$ subject to $y = D\hat{x}$.

Proof.

$$\left(\max_{x\in\mathcal{N}(D),\, x\neq 0} \frac{\overbrace{\sum_{k\in S}|x_k|}^{\text{1-norm} \Rightarrow \text{0-norm}}}{\underbrace{\sum_k |x_k|}_{\text{1-norm} \Rightarrow \text{0-norm}}} \approx \frac{S}{2S+1} < 1/2 \right.$$

$$\frac{S}{2S+1} < 1/2$$

$$\left. S < S + 1/2 \right)$$

back to the proof. We need to prove that for all $\alpha \in \mathcal{N}(D), \alpha\neq 0$,

$$\underbrace{\sum_k |x_k+\alpha_k|}_{\|x+\alpha\|_1} > \underbrace{\sum_k |x_k|}_{\|x\|_1}.$$

$$\|x+\alpha\|_1 > \|x\|_1$$

$$\left(\|x+\alpha\|_0 > \|x\|_0 \right)$$

$$|a+b| > |a| - |b|$$

$$\sum_k |x_k+\alpha_k| = \sum_{k\notin S} |x_k+\alpha_k| + \sum_{k\in S} |x_k+\alpha_k|$$

$$= \sum_{k\notin S} |\alpha_k| + \sum_{k\in S} |x_k+\alpha_k|$$

$$\geq \sum_{k\notin S} |\alpha_k| + \cancel{\sum_{k\in S} |x_k|} - \sum_{k\in S} |\alpha_k|$$

$$> \sum_k |x_k| = \cancel{\sum_{k\in S} |x_k|}$$

$$\Rightarrow \sum_{k\notin S} |\alpha_k| > \sum_{k\in S} |\alpha_k| \Big| + \sum_{k\in S} |\alpha_k|$$

$$\sum_k |\alpha_k| > 2 \sum_{k\in S} |\alpha_k|$$

$$\frac{\sum_{k\in S} |\alpha_k|}{\sum_k |\alpha_k|} < 1/2.$$

Theorem 3.5. (P1) applied to $y = Dx$ recovers x if

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right)$$

Proof. $(1+\mu(D))|\alpha_\ell| \leq \mu(D)\, \|\alpha\|_1$, for all $\ell = 1,\dots,n$

$\sum \quad \ell \in S$

$$(1+\mu(D)) \sum_{\ell\in S} |\alpha_\ell| \leq \mu(D) \|\alpha\|_1 \, |S|$$

$$(1+\mu(D))\, P_1(S,D) \leq \mu(D)\, |S|$$

$$P_1(S,D) \leq \frac{1}{1+1/\mu(D)}\, |S|$$

if $\frac{1}{1+1/\mu(D)} |S| < 1/2$

$$|S| < 1/2 \left(1+1/\mu(D)\right). \quad \text{q.e.d.}$$

Theorem 3.8. Let $D \in \mathbb{C}^{m\times n}$, $m \le n$, be a dictionary with coherence $\mu(D)$. Then

$$\mu(D) \ge \sqrt{\frac{n-m}{m(n-1)}}.$$

Proof. $G = D^H D \in \mathbb{C}^{n\times n}$, $D = [d_1\ d_2 \ldots d_n]$

1. G has ones along its main diagonal
2. G is pos. semi-def. with rank (at most) m

$\lambda = (\lambda_1, \ldots, \lambda_m)^T$ vector of nonzero eigenvalues of G.

$$Tr(G) = \sum_{i=1}^{m} \lambda_i = \|\lambda\|_1 = n$$

$$Tr(GG^H) = \|G\|_2^2 = \sum_{i=1}^{m} \lambda_i^2 = \|\lambda\|_2^2$$

$$\underbrace{\left(\frac{1}{m}\sum_{i=1}^{m}\lambda_i\right)^2}_{(EX)^2} \le \frac{1}{m}\sum_{i=1}^{m}\lambda_i^2, \text{ Jensen's inequality}$$

$$(EX)^2 \le EX^2$$

$$\frac{1}{m^{\cancel{2}}}\|\lambda\|_1^2 \le \frac{1}{\cancel{m}}\|\lambda\|_2^2$$

$$\|\lambda\|_1^2 \le m\|\lambda\|_2^2$$

$$\|G\|_2^2 \ge \frac{n^2}{m}$$

$$\|G\|_2^2 = n + \sum_{i=1}^{n}\sum_{j\ne i} |\langle d_i, d_j\rangle|^2 \ge \frac{n^2}{m}$$

$$n + n(n-1)\mu^2(D) \ge \frac{n^2}{m}$$

$$\mu^2(D) \ge \frac{n-m}{m(n-1)}$$

$$\mu(D) \geq \sqrt{\frac{n-m}{m(n-1)}}$$

$m \leq n$

for $m \ll n$, we get

$$\mu(D) \gtrsim \sqrt{\frac{\cancel{n}}{m(\cancel{n-1})}} \approx \frac{1}{\sqrt{m}}$$

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right) \approx \frac{1}{2}\left(1 + \sqrt{m}\right) \approx \frac{1}{2}\sqrt{m}$$

$$s \sim \sqrt{m}$$

$$\boxed{m \sim s^2}$$

square-root bottleneck

DFT-matrix, because of Vandermonde structure $\Rightarrow m \geq 2s$

3 regimes:

- $m \geq 2s$ linear
- $m \sim s \log n$ prob.
- $m \sim s^2$ square-root bottleneck