$$\mu(D) \geq \sqrt{\frac{n-m}{m(n-1)}}$$

$m \leq n$

for $m \ll n$, we get

$$\mu(D) \gtrsim \sqrt{\frac{\cancel{n}}{m(\cancel{n-1})}} \approx \frac{1}{\sqrt{m}}$$

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right) \approx \frac{1}{2}\left(1+\sqrt{m}\right) \approx \frac{1}{2}\sqrt{m}$$

$$s \sim \sqrt{m}$$

$$\boxed{m \sim s^2}$$

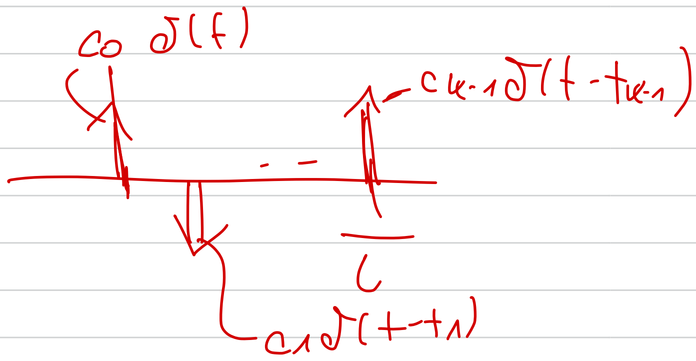
square-root bottleneck

DFT-matrix, because of Vandermonde structure $\Rightarrow m \geq 2s$

3 regimes:
- $m \geq 2s$ linear
- $m \sim s \log n$ prob.
- $m \sim s^2$ square-root bottleneck

4. Finite Rate of Innovation

$$x(t) = \sum_{k=0}^{K-1} c_k \, \delta(t - t_k), \quad t \in \mathbb{R}$$



$\delta(t)$ ... Dirac delta distribution

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-i2\pi f t}\, dt = e^{-i2\pi f t}\Big|_{t=0} = \underline{\underline{1}}$$

$$c_k \delta(t-t_k) \xrightarrow{\mathcal{F}} c_k e^{-i2\pi t_k f} = |c_k|\, e^{-i2\pi t_k f}$$

$$x = |x| e^{i\varphi(x)}$$

$$\varphi(\cdot) = -i2\pi t_k f \Rightarrow t_k = -\frac{\varphi(\cdot)}{2\pi i f}$$

Fourier series expansion of $x(t)$, assuming that $x(t) = x(t+\tau)$

$$d_n = \frac{1}{\tau}\int_0^\tau x(t)\, e^{-i2\pi n \frac{t}{\tau}}\, dt$$

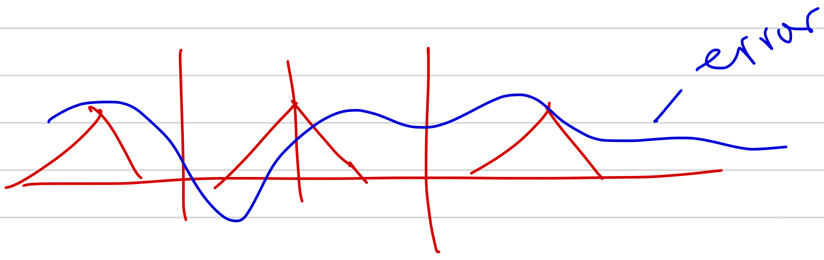
$$= \frac{1}{\tau}\int_0^\tau \sum_{k=0}^{K-1} c_k \delta(t-t_k)\, e^{-i2\pi n \frac{t}{\tau}}\, dt$$

$$= \frac{1}{\tau}\sum_{k=0}^{K-1} c_k \int_0^\tau \delta(t-t_k)\, e^{-i2\pi n \frac{t}{\tau}}\, dt$$

$$= \frac{1}{\tau}\sum_{k=0}^{K-1} c_k\, e^{-i2\pi n \frac{t_k}{\tau}}, \quad n \in \mathbb{Z}$$

Berlekamp–Massey

Peterson–Gorenstein–Zierler



$$+ \sum_k c_k \delta(t-t_k)$$

$$\downarrow$$

$$\sum_k c_k e^{-i2\pi t_k f}$$

"annihilating filter method"

starts with the construction of a filter

$$A(z) = \sum_{m=0}^{K} a_m z^{-m}$$

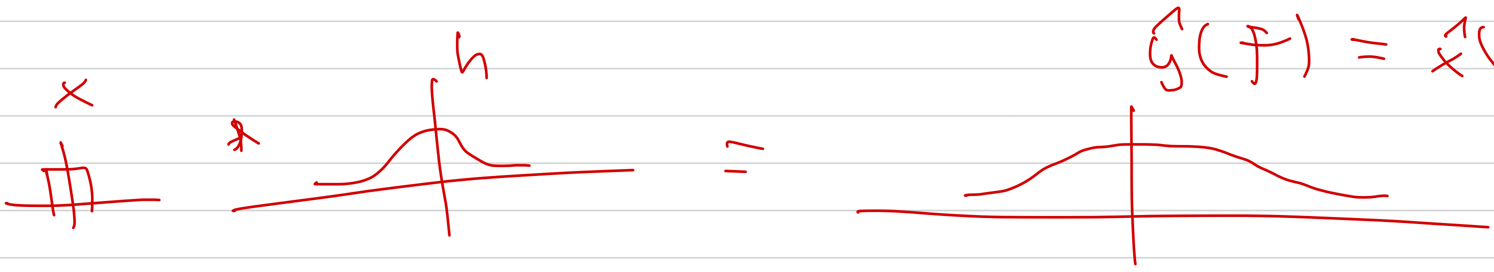
$$\hat{y}(\theta) = \hat{x}(\theta)\hat{h}(\theta) \qquad \underbrace{\sum_n x_n e^{-i2\pi\theta n}}_{z^{-n}}$$

convolution operation

$$x_n \longrightarrow \boxed{h_n} \longrightarrow y_n = \sum_{l=-\infty}^{\infty} x_l h_{n-l} = \sum_{l=-\infty}^{\infty} h_l x_{n-l}$$

filter

$$\left(y(t) = (x*h)(t) = \int x(\tau)\, h(t-\tau)\, d\tau \right)$$
$$\hat{y}(f) = \hat{x}(f)\,\hat{h}(f)$$



Z-transform : $a_n,\ n \in \mathbb{Z}$

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

$$p(z) = \underbrace{(z-1)(z-2)}_{\text{product form}} = z^2 - 2z - z + 2 = \underbrace{z^2 - 3z + 2}_{A(z)\ \ldots}$$

$$A(z) = \prod_{i=0}^{K-1} \left(1 - e^{-i2\pi \frac{t_k}{\tau}} z^{-1}\right)$$

a_n

$$\mathcal{Z}(\delta[n]) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \underline{\underline{1}}$$

$$1 - e^{-i2\pi t_k/\tau} z^{-1} \quad \bullet\!\!-\!\!\circ \quad \delta[n] - e^{-i2\pi \frac{t_k}{\tau}} \delta[n-1]$$

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

$$\sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{n'=-\infty}^{\infty} x[n'] z^{-(n'+1)} = z^{-1} X(z)$$

$$n' = n-1 \Rightarrow n = n'+1$$

$$e^{-i2\pi \frac{t_k}{\tau} n} * \left(\delta[n] - e^{-i2\pi \frac{t_k}{\tau}} \delta[n-1]\right) = e^{-i2\pi \frac{t_k}{\tau} n} - e^{-i2\pi \frac{t_k}{\tau} \cdot 1} e^{-i2\pi \frac{t_k}{\tau}(n-1)}$$

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$= e^{-i2\pi \frac{t_k}{\tau} n} - e^{-i2\pi \frac{t_k}{\tau}} e^{i2\pi \frac{t_k}{\tau}} e^{-i2\pi \frac{t_k}{\tau} n}$$

$$\underline{\underline{= 0}}$$

$$d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k\, e^{-i2\pi \frac{t_k}{\tau}} * \left(\left(\delta[n] - e^{-i2\pi \frac{t_k}{\tau}} \delta[n-1]\right) * (\ \sim\ ) \right)$$

$$* \left(\;\text{——}\; \right) \Big) = 0$$

$$d_n * a_n = 0$$

4.1. Finding the annihilating filter

$$d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k u_k^n$$

$$d_n * a_n = \sum_{l=0}^{K} a_l d_{n-l} = 0$$

$$S \in \mathbb{C}^{(K+1)\times(K+1)}$$

$$\begin{array}{c} n=0 \rightarrow \\ n=1 \rightarrow \\ \vdots \\ n=K \rightarrow \end{array} \begin{bmatrix} d_0 & d_{-1} & \cdots & d_{-K} \\ d_1 & d_0 & \cdots & d_{-K+1} \\ \vdots & & & \vdots \\ d_K & d_{K-1} & \cdots & d_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix} = 0$$

Hankel matrix

$2K+1$ Fourier series coefficients

Finding the t_k's & c_k's

$$A(z) = \sum_{m=0}^{K} a_m z^{-m} = \prod_{i=0}^{K-1} \left(1 - e^{-i2\pi \frac{t_k}{\tau}} z^{-1}\right)$$

!!! to be resolved

Zeros of $A(z)$ are given by $1 - e^{-i2\pi \frac{t_k}{\tau}} z^{-1} = 0$

$$e^{i2\pi \frac{t_k}{\tau}} = z^{-1}$$

$$z_k = e^{-i2\pi \frac{t_k}{\tau}}$$

$$t_k = -\frac{\tau}{2\pi} \arg(z_k)$$

4.1.2. Finding the c_k's

$$d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{e^{-i2\pi \frac{t_k}{\tau} n}}_{u_k^n}, \qquad u_k = e^{-i2\pi \frac{t_k}{\tau}}$$

$$\frac{1}{\tau} \underbrace{\overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & \cdots & u_{K-1} \\ u_0^2 & u_1^2 & \cdots & \\ u_0^{K-1} & u_1^{K-1} & \cdots & u_{K-1}^{K-1} \end{bmatrix}}^{F}}_{K \times K} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{bmatrix}}_{K \times 1} = \underbrace{\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{K-1} \end{bmatrix}}_{K \times 1}$$

Summary of the algorithm

Given $x(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k)$

1. Compute the Fourier series coefficients of $x(t)$
   $\Rightarrow d_n$
2. Out of the d_n, construct the matrix S
3. Find the right-singular vector corr to the smallest (=0 ideally) singular value
   $\Rightarrow a_n$
4. Find the zeros of the polynomial $A(z) = \sum_{m=0}^{K} a_m z^{-m}$
5. With these zeros, denoted as z_k, compute

$$\hat{t}_k = -\frac{\tau}{2\pi} \arg(z_k)$$

6. with these estimates for t_ℓ, i.e., with the $\hat{t}_\ell$, construct F and solve for c

Dimension of null-space of S

$$S = \frac{1}{c} \sum_{\ell=0}^{K-1} c_\ell \underbrace{\begin{bmatrix} 1 & u_\ell^{-1} & \cdots & u_\ell^{-K} \\ u_\ell & 1 & & \\ u_\ell^2 & & & \\ \vdots & & & \\ u_\ell^K & & & \end{bmatrix}}$$

$$= \frac{1}{c} \sum_{\ell=0}^{K-1} c_\ell \underbrace{\begin{bmatrix} 1 \\ u_\ell \\ u_\ell^2 \\ \vdots \\ u_\ell^K \end{bmatrix} \begin{bmatrix} 1 & u_\ell^{-1} & u_\ell^{-2} & \cdots \end{bmatrix}}_{\text{rank-1 matrix}}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & & u_{K-1} \\ u_0^2 & u_1^2 & & u_{K-1}^2 \\ \vdots & \vdots & & \vdots \end{bmatrix} \Rightarrow \text{full-rank}$$

$\Rightarrow$ Uniqueness

Chapter 6. The ESPRIT Algorithm

Paulraj, Roy, Kailath, 1986

$$x_n = \sum_{k=1}^{K} \alpha_k z_k^{\,n} \quad , \quad n \in \{0, 1, \dots, N-1\}$$

$$N \geq 2K$$

$$\text{cf.} \quad d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{e^{-i2\pi \frac{t_k}{\tau} n}}_{u_k^{\,n}}$$

$$z_k = e^{-d_k} \, e^{i2\pi f_k} \quad , \; d_k > 0$$

$$|z_k| \leq 1$$

$$z_k^{\,n} = \left(e^{-d_k}\right)^n e^{i2\pi f_k n}$$

$$V_{L \times K} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & & z_K \\ z_1^2 & z_2^2 & & z_K^2 \\ \vdots & \vdots & & \vdots \\ z_1^{L-1} & z_2^{L-1} & & z_K^{L-1} \end{pmatrix}$$

$$L \geq K$$

$$\mathcal{H}_L(x_0, x_1, \dots, x_{N-1}) = \begin{pmatrix} x_0 & x_1 & \dots & x_{N-L-1} & x_{N-L} \\ x_1 & x_2 & \dots & & x_{N-L+1} \\ x_2 & \dots & & & \\ \vdots & & & & \\ x_{L-1} & \dots & & \dots & x_{N-1} \end{pmatrix}$$

$$1 \leq L \leq N$$

6.2. Signal & Noise Subspaces

$$X = \mathcal{H}_L(x_0, \dots, x_{N-1})$$

$$K+1 \le L \le N-K-1$$

$$X = V_L D_\alpha V_{N-L+1}^T \in \mathbb{C}^{L \times (N-L+1)}$$

$$D_\alpha = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \phi \\ & \phi & \ddots & \\ & & & \alpha_K \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ z_1^2 & z_2^2 & \cdots \\ \vdots & \vdots & \\ z_1^{L-1} & z_2^{L-1} & \cdots \end{pmatrix} \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_K \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{N-L+1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{N-L+1} \\ & & \vdots & & \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots \\ \alpha_1 z_1 & \alpha_2 z_2 & \\ \alpha_1 z_1^2 & \alpha_2 z_2^2 & \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \\ \vdots & \vdots & \vdots & \cdots \\ 1 & z_K & z_K^2 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{K} \alpha_k & \sum_{k=1}^{K} \alpha_k z_k \\ \sum_{k=1}^{K} \alpha_k z_k & \end{pmatrix}$$

$$\underset{L \times (N-L+1)}{X} = \overset{L \times K}{V_L} \overset{K \times K}{D_\alpha} \overset{K \times (N-L+1)}{D_{N-L+1}^T}$$

$$\mathrm{SVD}(X) = \underbrace{\begin{pmatrix} \overset{L\times K}{S} & S_\perp \end{pmatrix}}_{U}^{L\times L} \overset{L\times(N-L+1)}{\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}} \overset{(N-L+1)\times(N-L+1)}{\begin{pmatrix} R^H \\ \underbrace{R_\perp^H} \end{pmatrix}}_{W^H}$$

Sylvester's inequality

$$= S \Delta R^H$$

$$\mathrm{rank}(A) + \mathrm{rank}(B) - n \leq \mathrm{rank}(\underset{m\times n}{A}\,\underset{n\times k}{B}) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$$

$$K + K - K \leq \mathrm{rank}(V_L D_\alpha) \leq K \Rightarrow \mathrm{rank}(V_L D_\alpha) = K$$

$$K+1 \leq L \leq N-K-1$$

$$K \leq N - L - 1$$

$$\mathrm{rank}(X) = K$$

$$L \times (N - L + 1)$$