$$\mathrm{SVD}(X) = \underbrace{\left(\overset{L\times k}{S} \quad S_\perp \right)}_{U} \overset{L\times L}{} \overset{L\times(N-L+1)}{\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}} \overset{(N-L+1)\times(N-L+1)}{\underbrace{\begin{pmatrix} R^H \\ R_\perp^H \end{pmatrix}}_{W^H}}$$

$$= S \Delta R^H$$

1. $\mathrm{span}(S) = \mathrm{span}(V_L)$

Sylvester's inequality

$$\mathrm{rank}(A) + \mathrm{rank}(B) - n \le \mathrm{rank}(\underset{m\times n}{A}\ \underset{n\times k}{B}) \le \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$$

$$k + K - K \le \mathrm{rank}(V_L D_\alpha) \le K \Rightarrow \mathrm{rank}(V_L D_\alpha) = K$$

$$k+1 \le L \le N-K-1$$

$$K \le N-L-1$$

$$\mathrm{rank}(X) = K$$

$$L \times (N-L+1)$$

2. $V_\uparrow = V_\downarrow D_z$, $D_z = \begin{pmatrix} z_1 & & \emptyset \\ & z_2 & \\ \emptyset & & \ddots \\ & & & z_K \end{pmatrix}$

The algorithm

$$\underset{L\times k}{S} = \underset{L\times k}{V_L} \underset{\substack{k\times K \\ r(P)=K}}{P}$$

$$S_\downarrow, S_\uparrow$$

$$S_\downarrow = V_\downarrow P \quad | \cdot P^{-1} \Rightarrow V_\downarrow = S_\downarrow P^{-1}$$

$$S_\uparrow = V_\uparrow P \Rightarrow V_\uparrow = S_\uparrow P^{-1}$$

$$V_\uparrow = V_\downarrow D_z$$

$$S_\uparrow P^{-1} = S_\downarrow P^{-1} D_z \quad | \cdot P$$

$$S_\uparrow = S_\downarrow \underbrace{P^{-1} D_z P}_{\Phi} \quad | \cdot S_\downarrow^{+}$$

$$\underline{\Phi = S_\downarrow^{+} S_\uparrow}$$

Similarity principle $\lambda_i(\Phi) = \lambda_i(D_z) = z_i$ ✓

1. form the Hankel data matrix X
2. $SVD(X) \Rightarrow S$
3. $S_\downarrow, S_\uparrow$
4. $\Phi = S_\downarrow^+ S_\uparrow$
5. $\lambda_i(\Phi) = z_i$

Review of the similarity principle

Def. 6.1. The matrices $X \in \mathbb{C}^{n\times n}$ and $Y \in \mathbb{C}^{n\times n}$ are similar if there exists an invertible $n\times n$ matrix P s.t. $X = P^{-1} Y P$.

Thm. 6.2. Let A and B be similar matrices. Then, A and B have the same eigenvalues with the same geometric multiplicities.

Proof.

$$A = P^{-1} B P \Rightarrow B = P A P^{-1}$$

$Au = \lambda u$ $\qquad\qquad$ $Bu = \lambda u$

$P^{-1} B P u = \lambda u \quad | P\cdot$ $\qquad\qquad$ $P A P^{-1} u = \lambda u$

$B P u = \lambda P u$ $\qquad\qquad$ $A \underbrace{P^{-1} u}_{u'} = \lambda \underbrace{P^{-1} u}_{u'}$. q.e.d.

full rank of $S_\downarrow = V_{L-1} P$ — $r(P) = K$

$r(V_{L-1}) = K$

$L \geq K+1$

$$\underbrace{r(V_{L-1})}_{K} + \underbrace{r(P)}_{K} - K \leq r(S_\downarrow) = r(V_{L-1} P) \leq \min\{\underbrace{r(V_{L-1})}_{K}, \underbrace{r(P)}_{K}\}$$

$$\Rightarrow r(S_\downarrow) = K$$

6.4. Finding the zeros of a polynomial through ESPRIT

$$p(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_k z^k$$

$$p(z) = (\alpha_0 \ \alpha_1 \ \dots \ \alpha_k) \begin{pmatrix} 1 \\ z \\ \vdots \\ z^k \end{pmatrix}$$

denote the zeros of $p(z)$ by $z_0, z_1, \dots, z_{k-1}$

$$\underbrace{(\alpha_0 \ \alpha_1 \ \dots \alpha_k)}_{\alpha} \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_{k-1} \\ \vdots & \vdots & & \vdots \\ z_0^k & z_1^k & \dots & z_{k-1}^k \end{pmatrix}}_{V \in \mathbb{C}^{(k+1)\times k}} = 0^T$$

Vandermonde

$$\alpha = \underset{1\times(k+1)}{[\overset{1\times 1}{u_1} \ u_2]} \begin{bmatrix} \sigma & 0_{1\times k} \\ 0_{k\times 1} & 0_{k\times k} \end{bmatrix} \begin{bmatrix} w^T \\ Z^T \end{bmatrix} \begin{matrix} 1\times(k+1) \\ k\times(k+1) \end{matrix}$$

$$= \underbrace{u_1 \sigma}_{1\times 1} \underset{1\times(k+1)}{w^T}$$

$$\alpha Z = 0_{1\times k}$$

$$\underset{(k+1)\times k}{V} = \underset{(k+1)\times k}{Z} \overset{k\times k}{T}$$

$$V_{\downarrow} = Z_{\downarrow} T$$

$$V_{\uparrow} = Z_{\uparrow} T$$

$$V_{\uparrow} = V_{\downarrow} D_z$$

$$Z_{\uparrow} T = Z_{\downarrow} T D_z$$

$$Z_{\uparrow} = Z_{\downarrow} \underbrace{T D_z T^{-1}}_{\Phi}$$

$$\Phi = z_0 + z_1$$

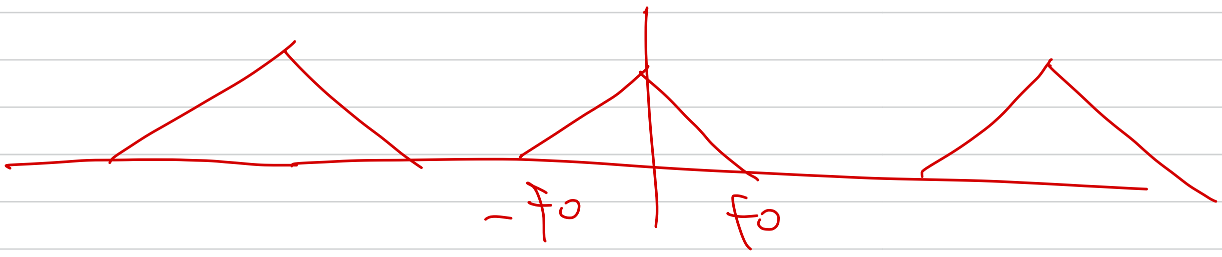
$$d_i(\Phi) = z_i \quad \checkmark$$

Chapter 5

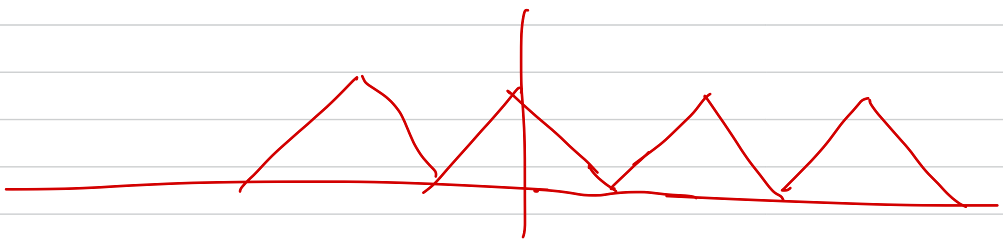
Sampling of multi-band signals

Classical sampling theorem

oversampling

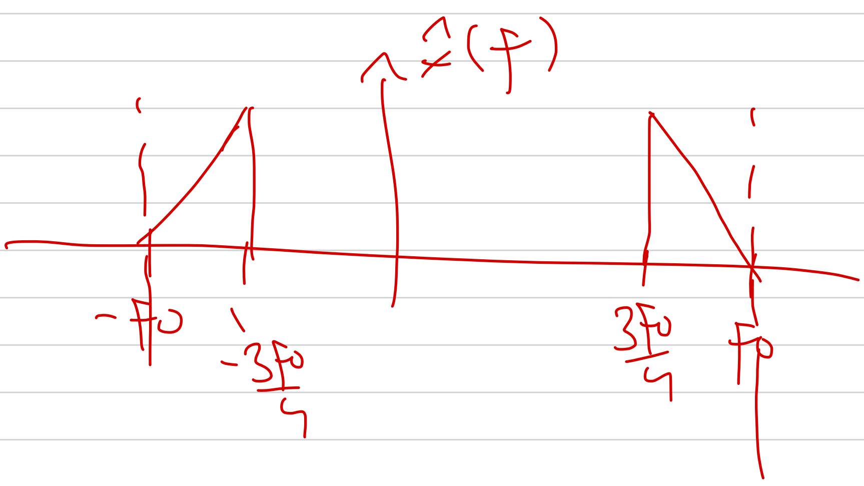


critical sampling



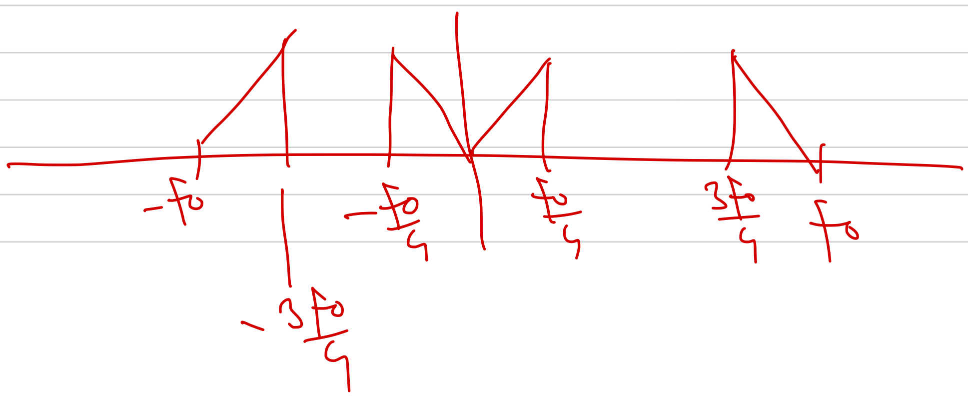
undersampling

Spectrally sparse signals

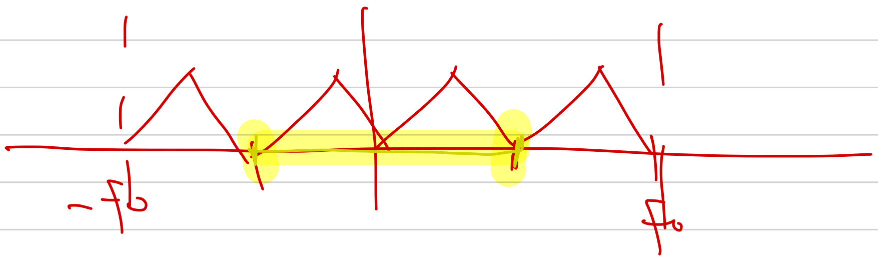


eff. bw. $\frac{f_0}{2}$

Nyquist rate $= 2f_0$



$f_s = f_0$



$f_s = \frac{f_0}{2}$

Multi-band signals

Consider a signal x with spectral occupancy $\mathcal{I} \in [-f_0, f_0]$

$|\mathcal{I}| = \sum \ldots$

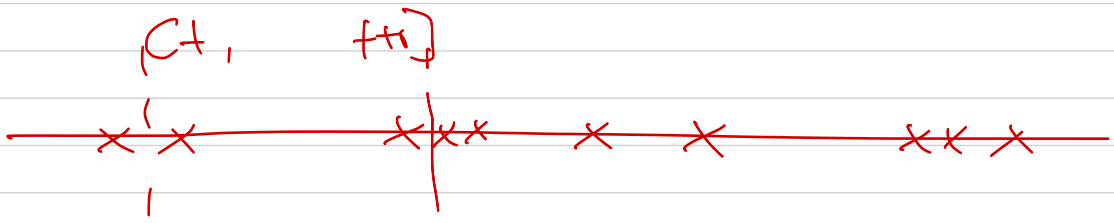
Theorem 5.1 (Landau, 1967). To reconstruct stably, we need

$$D^-(P) = \lim_{r \to \infty} \inf_{t \in \mathbb{R}} \frac{|P \cap [t, t+r]|}{r} \geq |\mathcal{I}|,$$

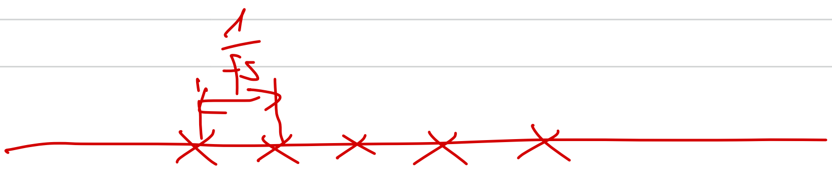
where $D^-(P)$ denotes the lower Beurling density of the sampling set $P = \{t_n\}$.

5.2.1. Interpretation of the lower Beurling density

1. Fix r
2. Slide a window of length r across the real axis, i.e., consider the interval $[t, t+r]$ and intersect it with $P = \{t_n\}$. Find the smallest no. of sampling points contained in any such interval and divide this no. by r.
3. Take the window length to infinity and compute the limit of the quantity in 2. as $r \to \infty$.



regular sampling



$f_s = \frac{1}{T}$

$$\frac{r}{1/F_s} = r f_s \Rightarrow \frac{r f_s}{r} = f_s$$

Stable reconstruction

Def. 5.2. A set of points $P = \{t_n\}$ is called a stable sampling set if for all $x_1, x_2 \in \mathcal{H}$,

$$A \|x_1 - x_2\|_{\mathcal{H}}^2 \le \|\underbrace{x_1(P)}_{\{x_1(t_n)\}_{n\in\mathbb{Z}}} - x_2(P)\|_2^2 \le B \|x_1 - x_2\|_{\mathcal{H}}^2$$

for some $A > 0$ and $B < \infty$.

now, consider a set of signals S that form a vector space

$$A \|x\|_{\mathcal{H}}^2 \le \|x(P)\|_2^2 \le B\|x\|_{\mathcal{H}}^2, \quad \forall x \in S$$

$x_1 - x_2 \in S$ given that $x_1, x_2 \in S$

$$\langle Tx, Tx\rangle = \langle x, \overbrace{T^*T}^{S} x\rangle$$

$$A \|x\|_{\mathcal{H}}^2 \le \|Tx\|_2^2 \le B \|x\|_{\mathcal{H}}^2$$

$$\langle Sx, x\rangle$$

$$S = T^*T$$

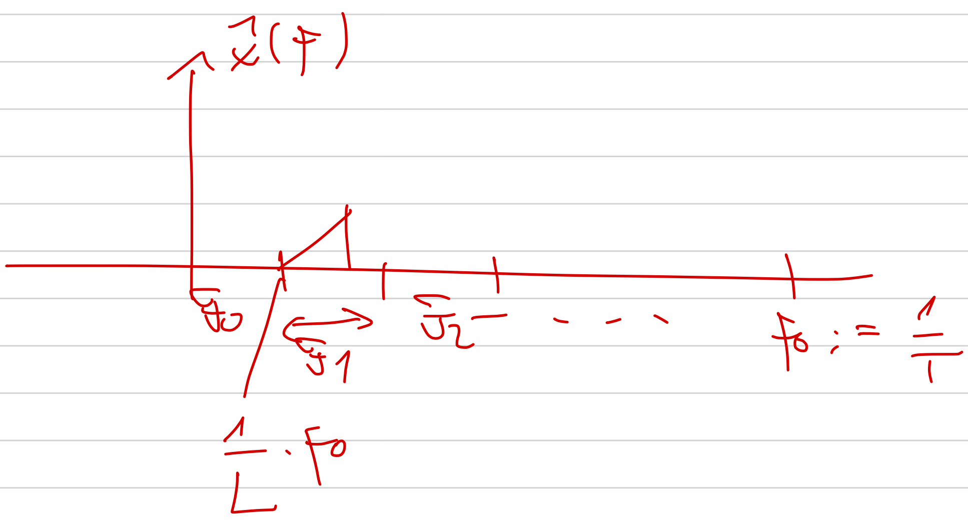
go back to spectrum-blind sampling

$$B(\mathcal{I}) := \{x \in \mathcal{L}^2(\mathbb{R}) : \hat{x}(f) = 0, \forall f \notin \mathcal{I}\}.$$

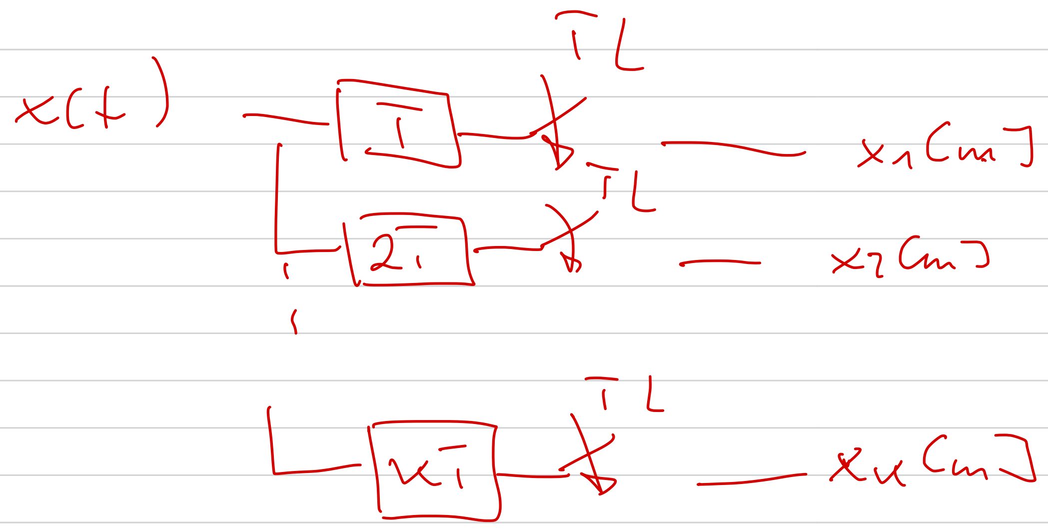
5.3. Multicoset sampling

partition the overall support set into L cells $\mathcal{F}_i$ of equal length $\frac{f_0}{L}$, i.e.,

$$\mathcal{F}_i := [i \tfrac{f_0}{L}, (i+1)\tfrac{f_0}{L}), \quad i \in \{0, \dots, L-1\}$$

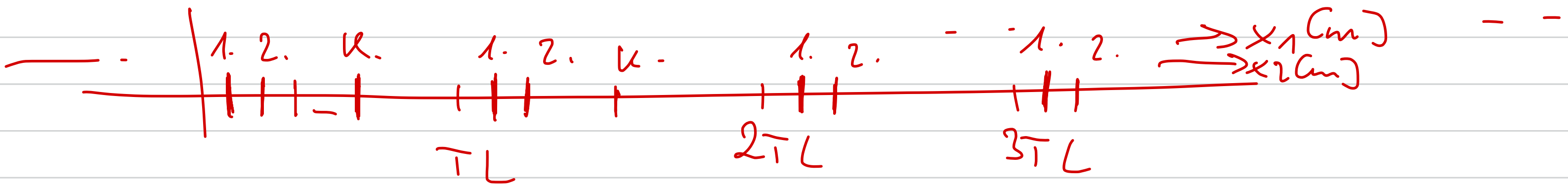


$$|\mathcal{I}| \approx \frac{s f_0}{L} = \frac{s}{TL}$$



$$x_\ell[m] := x((mL+\ell)T),\ m \in \mathbb{Z}$$

$$\mathcal{D}(P) = K \cdot \frac{1}{TL} \overset{\text{want}}{=} \frac{S}{TL}$$



$$K < L$$

$$x_d^{(\ell)}(f) = \sum_{m \in \mathbb{Z}} x_\ell[m]\, e^{-i2\pi f m T L}, \qquad f \in [0,1)$$

$$= \sum_{m \in \mathbb{Z}} x((mL+\ell)T)\, e^{-i2\pi f m T L}$$

$$= e^{i2\pi f \ell T} \sum_{m \in \mathbb{Z}} x((mL+\ell)T)\, e^{-i2\pi f (mTL+\ell T)}$$

$$\overset{\text{Poisson s.f.}}{=} e^{i2\pi f \ell T} \frac{1}{TL} \sum_{m \in \mathbb{Z}} \hat{x}\left(f + \frac{m}{TL}\right) e^{i2\pi \frac{m\ell}{L}}$$

Poisson s.f.:

ℓT

$$\sum_{\ell \in \mathbb{Z}} s(t+\ell T) = \frac{1}{T} \sum_{\ell} \hat{s}\left(\frac{\ell}{T}\right) e^{i2\pi \frac{\ell}{T} t}$$

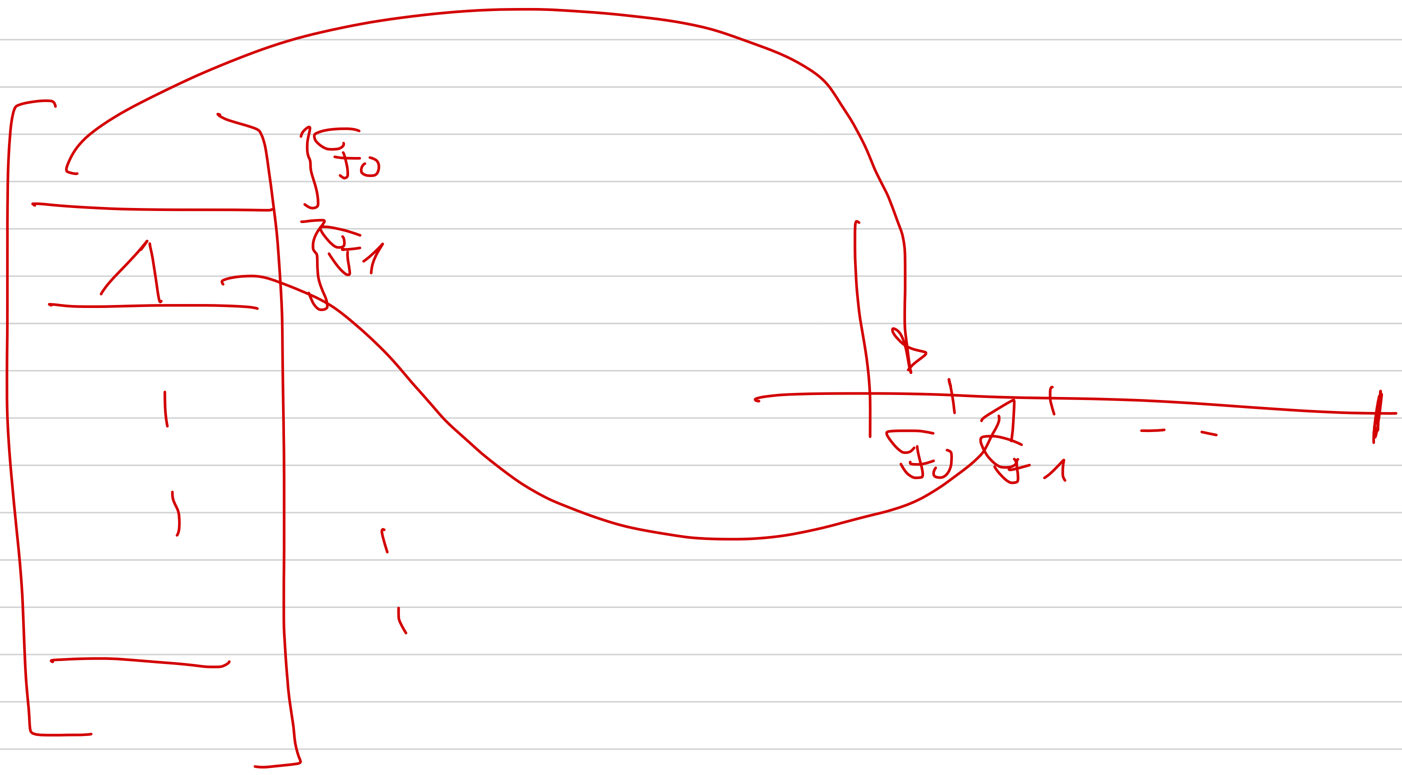
apply P.S.F. to $x(t)\, e^{-i2\pi f_0 t}$ $\left(\circ\!\!-\!\!\bullet\ \hat{x}(f+f_0) \right)$

$$e^{i2\pi \frac{m}{L\cancel{T}} \ell \cancel{T}} = e^{i2\pi \frac{m\ell}{L}}$$

$$y_\ell(f) := x_d^{(\ell)}(f)\, e^{-i2\pi f T\ell}$$

$$= \sum_{m\in\mathbb{Z}} \hat{x}\left(f+\frac{m}{TL}\right) e^{i2\pi\frac{m\ell}{L}}, \quad f\in\left[0,\frac{1}{TL}\right)$$

$$\underbrace{\begin{bmatrix} y_1(f) \\ y_2(f) \\ \vdots \\ y_k(f) \end{bmatrix}}_{k \text{ measurements}} = \underbrace{\begin{bmatrix} 1 & e^{i2\pi\frac{1}{L}} & \cdots & e^{i2\pi\frac{L-1}{L}} \\ 1 & e^{i2\pi\frac{2}{L}} & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \vdots & & \vdots \end{bmatrix}}_{\text{Vandermonde}} \underbrace{\begin{bmatrix} \hat{x}(f) \\ \hat{x}\left(f+\frac{1}{TL}\right) \\ \vdots \\ \hat{x}\left(f+\frac{L-1}{TL}\right) \end{bmatrix}}_{S\text{-sparse}} \quad f\in\left[0,\frac{1}{LT}\right)$$



$f\in\left[0,\frac{1}{TL}\right)$

with $k=s$, we have to invert an $s\times s$ Vandermonde matrix with diff. nodes $\Rightarrow$ uniqueness

$$D^-(P) = \frac{k}{LT} = \frac{s}{LT} \approx |I|.$$

5.4. Spectrum-blind sampling

$$X(C) = \bigcup_{|I|\le C} B(I)$$

choose $k=2s$ $\Rightarrow$ $2s\times 2s$ Vand. matrix $\Rightarrow$ full rank $\Rightarrow$ injectivity

In summary, we found that $\mathcal{D}_T(p) \geq 2C$ is necessary & sufficient

proved only injectivity

For stable recovery -