In summary, we found that $\mathcal{D}T(P) \geq 2C$ is necessary & sufficient

proved only injectivity

for stable recovery.

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3 regimes

1. $m = cs$ — FRI / multi-band
2. $m = c' s^2$
   square-root bottleneck
   coherence-based recovery thresholds
3. $m = c'' s \log n$ ⟸ today

m ... # measurements
s ... sparsity level
n ... ambient dim.

$$y = \underbrace{\Phi}_{m \times n} \underbrace{x}_{n \times 1} + \underbrace{n}_{m \times 1}$$

n: noise

Def. For each integer $s = 1, 2, \ldots$, define the isometry constant δ_s of a matrix Φ as the smallest number s.t.

$$(1 - \delta_s) \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_s) \|x\|_2^2$$

holds for all s-sparse vectors x. A vector x is s-sparse if at most s of its entries are nonzero.

Theorem 7.2. Let $y = \Phi x$. Assume that $\delta_{2s} < \sqrt{2} - 1$. Then, the solution x^* to

$$\min_{\tilde{x} \in \mathbb{R}^n} \|\tilde{x}\|_1 \quad \text{subject to } \Phi \tilde{x} = y$$

obeys

$$\|x^* - x\|_1 \leq C_0 \|x - x_s\|_1$$

and

$$\|x^* - x\|_2 \leq C_0 s^{-1/2} \|x - x_s\|_1$$

for some C_0 specified in the proof of the statement. Here, x_s

is the vector obtained by setting all but the s largest (in $|\cdot|$) entries of x equal to zero.

Theorem 7.3. Let $y = \Phi x + n$. Assume that $\delta_{2s} < \sqrt{2} - 1$ and $\|n\|_2 \leq \varepsilon$. Then, the solution x^* to

$$\min_{\tilde{x} \in \mathbb{R}^n} \|\tilde{x}\|_1 \qquad \text{subject to} \qquad \|y - \Phi \tilde{x}\|_2 \leq \varepsilon$$

obeys

$$\|x^* - x\|_2 \leq C_0 s^{-1/2} \|x - x_s\|_1 + C_1 \cdot \varepsilon .$$

$\binom{n}{s}$ Subspaces

$\leq \left(\frac{en}{s}\right)^s \Rightarrow s \log\left(\frac{en}{s}\right)$

x_s -- covering

JL: set $\mathcal{U}$ of m points in $\mathbb{R}^n$
embed these points into $\mathbb{R}^k$, with $k < n$ while approximately preserving pairwise distances between the points.

The JL states that any set of m points can be embedded in $k = \Theta(\log(m)/\varepsilon^2)$ dimensions while the distances between the points change by a factor of at most $1 \pm \varepsilon$.

Lemma 8.1. (JL-Lemma). Choose ε with $\varepsilon \in (0,1)$ and suppose that k satisfies

$$k \geq \frac{8}{\varepsilon^2 - \varepsilon^3} \log(2m).$$

Then, for every set $\mathcal{U}$ of m points, there exists a map $f: \mathbb{R}^n \to \mathbb{R}^k$ s.t. for all $u, u' \in \mathcal{U}$, we have

$$(1-\varepsilon)\|u-u'\|^2 \leq \|f(u) - f(u')\|^2 \leq (1+\varepsilon)\|u-u'\|^2 . \qquad \text{(JL)}$$

Lemma 8.2. Let $A \in \mathbb{R}^{k \times n}$ be a random matrix with i.i.d. $\mathcal{N}(0, 1/k)$ entries. Then, for ε with $\varepsilon \in (0,1)$, and fixed $u \in \mathbb{R}^n$,

$$P\left(\left|\|Au\|^2 - \overbrace{E\left[\|Au\|^2\right]}^{\|u\|^2}\right| \geq \varepsilon \|u\|^2\right) < 2 e^{-k \frac{\varepsilon^2 - \varepsilon^3}{4}}$$

with

$$E\left[\|Au\|^2\right] = \|u\|^2 .$$

$$\left| \|Au\|^2 - \|u\|^2 \right| \geq \varepsilon \|u\|^2$$

$$\|Au\|^2 - \|u\|^2 \geq \varepsilon \|u\|^2$$

$$\underline{\|Au\|^2 \geq (1+\varepsilon) \|u\|^2}$$

$$-\|Au\|^2 + \|u\|^2 \geq \varepsilon \|u\|^2$$

$$\underline{\|Au\|^2 \leq (1-\varepsilon) \|u\|^2}$$

Lemma 8.2 $\Rightarrow$ JL Lemma

consider all $m(m-1)/2$ pairs $u, u' \in \mathcal{U}$

note that (JL) is violated with prob $2 m^2 e^{-k \frac{\varepsilon^2 - \varepsilon^3}{4}}$

$$2m^2 e^{-k \frac{\varepsilon^2-\varepsilon^3}{4}} < 1/2 \quad \leftarrow \text{prob. method}$$

$$k \geq \frac{4 \cdot 2}{\varepsilon^2 - \varepsilon^3} \log(2m)$$

Proof of the concentration inequality:

a_j^T is the j-th row of A and set $X_j = \frac{\sqrt{k}}{\|u\|} a_j^T u$

$$\operatorname{var}(X_j) = \frac{k}{\|u\|^2} \underbrace{\frac{1}{k}}_{\mathcal{N}(0,1/k)} \|u\|^2 = 1 .$$

$$X = \sum_{j=1}^{k} X_j^2 = \frac{k}{\|u\|^2} \sum_{j=1}^{k} |a_j^T u|^2$$

$$= \frac{k}{\|u\|^2} \|Au\|^2$$

X_j are i.i.d. $\mathcal{N}(0,1)$

$$P\left(\underbrace{\|Au\|^2}_{X \frac{\|u\|^2}{k}} \geq (1+\varepsilon)\|u\|^2\right) = P\left(X \geq (1+\varepsilon) k\right)$$

$$= P\left(e^{\lambda X} \geq e^{\lambda(1+\varepsilon)k}\right)$$

$$\overset{\text{Markov}}{\leq} \frac{1}{e^{(1+\varepsilon)k\lambda}} E[e^{\lambda X}]$$

$$= \frac{1}{e^{(1+\varepsilon)k\lambda}} \prod_{i=1}^{k} E[e^{\lambda X_i^2}]$$

$$\overset{\text{i.i.d.}}{=} \frac{1}{e^{(1+\varepsilon)k\lambda}} \left(E[e^{\lambda X_1^2}] \right)^k$$

$$E[e^{\lambda X_1^2}] = \frac{1}{\sqrt{1-2\lambda}}, \quad \lambda < 1/2.$$

$$P(\|Au\|^2 \geq (1+\varepsilon)\|u\|^2) \leq \left(\frac{e^{-2(1+\varepsilon)\lambda}}{1-2\lambda} \right)^{k/2}$$

min. RHS over λ

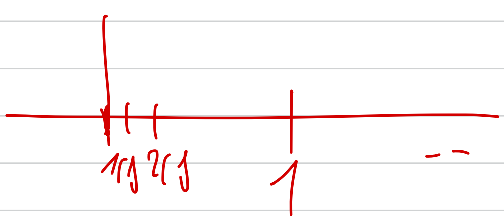
$$\lambda = \frac{\varepsilon}{2(1+\varepsilon)}$$

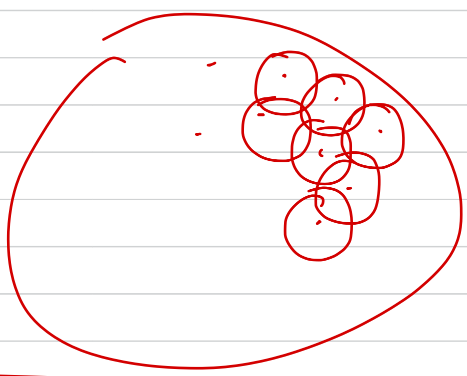
$$P(\|Au\|^2 \geq (1+\varepsilon)\|u\|^2) \leq \left((1+\varepsilon) e^{-\varepsilon} \right)^{k/2} < e^{-(\varepsilon^2-\varepsilon^3)\frac{k}{4}}$$

$$1+\varepsilon \leq e^{\varepsilon - \frac{\varepsilon^2-\varepsilon^3}{4}}$$

Taylor exp. of e^x

$$P(\|Au\|^2 \leq (1-\varepsilon)\|u\|^2) < e^{-(\varepsilon^2-\varepsilon^3)\frac{k}{4}}$$





Chapter 9. Verifying the Restr. Isom. Prop. through the JL Lemma

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Lemma 9.1. Let $\Phi \in \mathbb{R}^{m \times n}$ be an i.i.d. $\mathcal{N}(0, 1/m)$ matrix. Then, for any set S with $|S| = k < m$ and any $\delta \in (0,1)$, we have

$$(1-\delta)\|x\| \le \|\Phi x\| \le (1+\delta)\|x\|, \quad \forall x \in \mathcal{X}_S$$

with prob.

$$\ge 1 - 2(12/\delta)^k e^{-c_0(\delta/2)m}$$

where $c_0(x) = \frac{1}{4}(x^2 - x^3)$.

Proof. it suffices to consider $\|x\| = 1$

covering Q_S s.t. i) $Q_S \subseteq \mathcal{X}_S$, $\|q\| = 1$ for all $q \in Q_S$

ii) for all $x \in \mathcal{X}_S$ with $\|x\| = 1$, we have

$$\min_{q \in Q_S} \|x - q\| \le \delta/4 .$$

$\exists\, Q_S$ s.t. $|Q_S| \le (12/\delta)^k$.

$$(1 - \delta/2)\|q\|^2 \le \|\Phi q\|^2 \le (1 + \delta/2)\|q\|^2, \quad \forall q \in Q_S$$

holds w.p.

$$\ge 1 - 2(12/\delta)^k e^{-c_0(\delta/2)m}$$

$$c_0(x) = \frac{x^2 - x^3}{4} .$$

A is the smallest no. s.t.

$$\|\Phi x\| \le (1 + A)\|x\|, \quad \forall x \in \mathcal{X}_S$$

$x \in \mathcal{X}_S$ ($\|x\| = 1$) $\Rightarrow$ $q \in Q_S$ s.t. $\|x - q\| \le \delta/4$ (covering)

$$\|\Phi x\| = \|\Phi(q + x - q)\| \le \|\Phi q\| + \overbrace{\|\Phi(x-q)\|}^{(1+A)\delta/4}$$

$$\le \sqrt{1 + \delta/2}\,\|q\|^2 \le 1 + \delta/2$$

$$\le 1 + \delta/2 + (1 + A)\delta/4$$

$$\cancel{1} + A \le \cancel{1} + \delta/2 + (1+A)\delta/4$$

$$A \le \delta/2 + (1 + A)\delta/4$$

$$A \le \delta .$$

we therefore established that $\|\Phi x\| \le (1+\delta)\|x\|$

$$\|\Phi x\| \geq \|\Phi q\| - \|\Phi(x-q)\| \geq 1-\delta/2 - (1+\delta)\delta/4$$

$$\geq 1-\delta. \qquad \text{q.e.d.}$$

Theorem. Suppose that $\underset{m\times n}{m,n}$, and $\delta \in (0,1)$ are given. If the pdf generating Φ satisfies the concentration inequality in Lemma 8.2, then there exist constants $c_1, c_2 > 0$, depending only on δ s.t. the restr. isom. prop. holds for Φ with the prescribed δ and every $k \leq \frac{c_1 m}{\log(n/k)}$ w.p. $\geq 1 - 2e^{-c_2 m}$.

$$\underset{\uparrow \atop s}{k} \leq \frac{c_1 \overset{\swarrow \# \text{meas.}}{m}}{\log(\underset{\uparrow \atop \text{amb. dim.}}{n}/\underset{\nwarrow s}{k})} \quad \Rightarrow \quad \boxed{m \geq C\, s \log(n/s)}$$

Proof. We know that for each of the k-dim. subspaces X_S, the matrix Φ will fail to satisfy

$$(1-\delta)\|x\| \leq \|\Phi x\| \leq (1+\delta)\|x\|, \quad \forall x \in X_S \qquad (*)$$

w.p. $\leq 2(12/\delta)^k e^{-c_0(\delta/2)m}$.

Key idea: There are $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ such subspaces $\Rightarrow$ by the union bound $(*)$ will fail to hold w.p.

$$\leq \underbrace{2(en/k)^k}_{\substack{\uparrow \\ \text{counting} \\ \text{subspaces} \\ \text{or equiv.} \\ \text{no. of support sets}}} \underbrace{(12/\delta)^k}_{\nwarrow \text{ discretization of given } X_S} e^{-c_0(\delta/2)m}.$$

q.e.d.

$$y = W\underset{\uparrow \atop s\text{-sparse}}{x} \qquad \Phi y = \underset{\uparrow \atop \text{i.i.d.}}{\Phi} \overset{\swarrow \text{orth.}}{W} x$$