Thm. 10.8. $\quad r^{*,\mathrm{eff}}(C,D) \le r^*(C)$

Proof. effective best M-term approx. of $f \in C$

$$\|f - \sum_{i \in I_M} c_i \varphi_i\| \le C M^{-r}$$

$r < r^{*,\mathrm{eff}}(C,D)$

$|I_M| \le \pi(M)$

1. indices of participating dictionary elements

$\log_2 \pi(M) \le \log_2 M^P = P \log_2 M$

$M P \log_2 M = \underline{\underline{C M \log_2 M}} = \mathcal{O}(M \log_2 M)$

2. quantization of the c_i

$$\sum_{i \in I_M} c_i \varphi_i = \sum_{i \in \tilde{I}_{\tilde{M}}} \tilde{c}_i \tilde{\varphi}_i$$

$\tilde{M} \le M$

$$e = f - \sum_{i \in \tilde{I}_{\tilde{M}}} \tilde{c}_i \tilde{\varphi}_i = f - \sum_{i \in I_M} c_i \varphi_i$$

$$\underbrace{\|\sum_{i \in \tilde{I}_{\tilde{M}}} \tilde{c}_i \tilde{\varphi}_i\|} = \|f - e\| \le \|f\| + \|e\|$$

$$\sqrt{\sum_{i \in \tilde{I}_{\tilde{M}}} |\tilde{c}_i|^2} \le \underbrace{\sup_{f \in C} \|f\|}_{\le D} + C M^{-r} = \underline{\underline{D + C M^{-r} < \infty}}$$

quantization levels: $M^{-(r+1/2)} \mathbb{Z}$

no. of bits needed to encode one $Q(\tilde{c}_i) = \log_2 \dfrac{C'}{M^{-(r+1/2)}}$

$$= \log_2 C' M^{(n+1/2)}$$

$$\hat{c}_i = Q(\tilde{c}_i) \quad (\lfloor \ \rfloor, \lceil \ \rceil) \qquad = C'' \log_2 M$$

#bits needed to encode all $\hat{c}_i$ $= C'' M \log_2 M$

$$\underbrace{\| F - \sum_{i \in \hat{I}_M} \hat{c}_i \hat{\varphi}_i \|}_{\leq C M^{-n}} = \| f - \sum_{i \in \hat{I}_M} \tilde{c}_i \hat{\varphi}_i + \sum_{i \in \hat{I}_M} \tilde{c}_i \hat{\varphi}_i - \sum_{i \in \hat{I}_M} \hat{c}_i \hat{\varphi}_i \|$$

$$\leq \underbrace{\| f - \sum_{i \in \hat{I}_M} \tilde{c}_i \hat{\varphi}_i \|}_{= \|e\| \leq C M^{-n}} + \underbrace{\| \sum_{i \in \hat{I}_M} (\tilde{c}_i - \hat{c}_i) \hat{\varphi}_i \|}_{= \underbrace{\left(\sum_{i \in \hat{I}_M} |\tilde{c}_i - \hat{c}_i|^2 \right)^{1/2}}_{\leq C' M^{-n}}}$$

$$\sum_{i \in \hat{I}_M} |\tilde{c}_i - \hat{c}_i|^2 \leq \sum_{i \in \hat{I}_M} M^{-2n-1} \leq M \cdot M^{-2n-1} = M^{-2n}$$

$$|\hat{I}_M| \leq |I_M| \leq M$$

$$\| f - \sum_{i \in \hat{I}_M} \hat{c}_i \varphi_i \| \leq \underbrace{C M^{-n}}_{\varepsilon}$$

#bits $= O(M \log_2 M)$

1. recall that all arguments we used are valid for $n < n^{*,\mathrm{eff}}(C,D)$
2. #bits $= O(M \log_2 M) = O(\varepsilon^{-1/n} \log_2 \varepsilon^{-1/n})$

$$= O\left(\varepsilon^{-1/(n^{*,\mathrm{eff}}(C,D) - \delta)}\right)$$

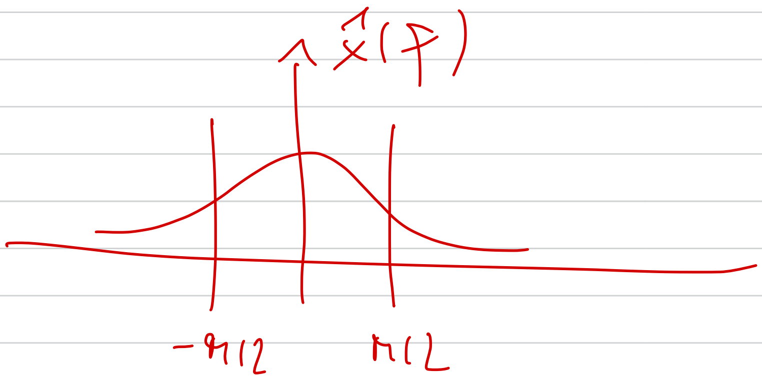
$$n^{*,\mathrm{eff}}(C,D) - \delta \leq n^*(C).$$

Def. 10.9.

$$n^{*,\mathrm{eff}}(C,D) = n^*(C)$$

C is optimally representable in $\mathcal{D}$

Chapter 11



11.1.1. Linear Approximation

Let $\mathcal{B} = \{g_j\}_{j \in \mathbb{N}_0}$ be an ONB for $\mathcal{H}$. Every $x \in \mathcal{H}$ can be written as

$$x = \sum_{j=0}^{\infty} \langle x, g_j \rangle g_j$$

approximate x by the first M terms, we get

$$x_M = \sum_{j=0}^{M-1} \langle x, g_j \rangle g_j$$

$$x - x_M = \sum_{j=M}^{\infty} \langle x, g_j \rangle g_j$$

$$\varepsilon_e[M] = \|x - x_M\|^2 = \Big\| \sum_{j=0}^{\infty} \langle x, g_j \rangle g_j - \sum_{j=0}^{M-1} \langle x, g_j \rangle g_j \Big\|^2$$

$$= \Big\| \sum_{j=M}^{\infty} \langle x, g_j \rangle g_j \Big\|^2 = \sum_{j=M}^{\infty} |\langle x, g_j \rangle|^2$$

$$\|x\|^2 < \infty$$

$$\sum_{j=0}^{\infty} |\langle x, g_j \rangle|^2 < \infty$$

$\lim_{M\to\infty} \mathcal{E}_e[M] = 0$? how fast

j^{2s} j^{-2} $= j^{1+\varepsilon-2}$ $= j^{-1+\varepsilon}$

Thm. 11.1. Let $s > 1/2$ and $\sum_{j=0}^{\infty} j^{2s} \underbrace{|\langle x, g_j\rangle|^2} < \infty$. There exist constants $A, B > 0$ such that

$j^{-s-1/2-\varepsilon} \Rightarrow j^{2s} \cdot j^{-2s-1-2\varepsilon}$

$$A \sum_{j=0}^{\infty} j^{2s} |\langle x, g_j\rangle|^2 \le \sum_{M=0}^{\infty} M^{2s-1} \mathcal{E}_e[M] \le B \sum_{j=0}^{\infty} j^{2s} |\langle x, g_j\rangle|^2$$

and hence $\mathcal{E}_e[M] = o(M^{-2s})$, i.e., $\lim_{M\to\infty} \mathcal{E}_e[M] M^{2s} = 0$.

$$\mathcal{E}_e[M] \sim M^{-2s}$$

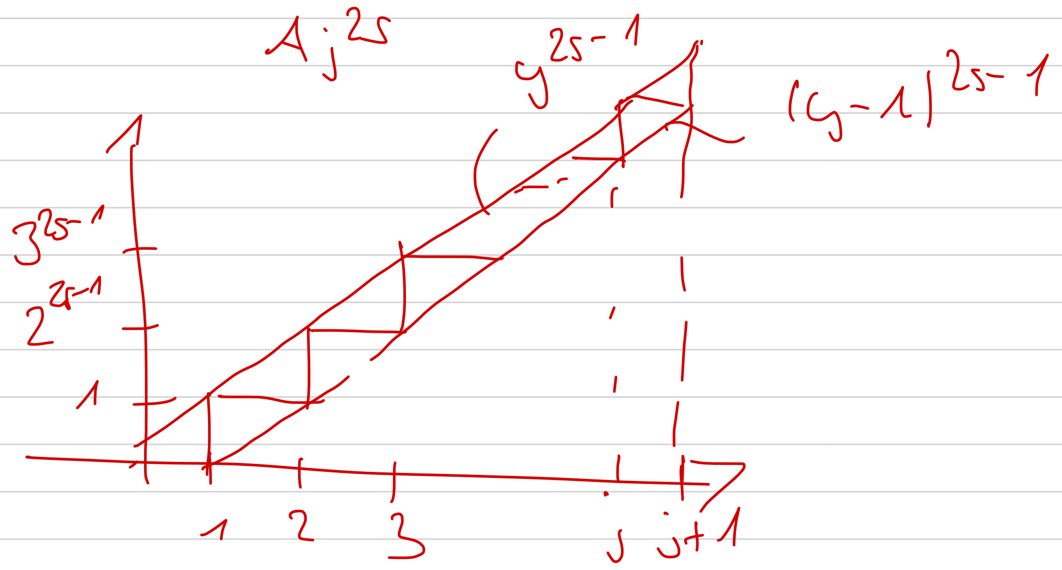
Proof.

$$\sum_{M=0}^{\infty} M^{2s-1} \mathcal{E}_e[M] = \sum_{M=0}^{\infty} M^{2s-1} \sum_{j=M}^{\infty} |\langle x, g_j\rangle|^2$$

$$= 1 \cdot \sum_{j=1}^{\infty} |\langle x, g_j\rangle|^2 + 2^{2s-1} \sum_{j=2}^{\infty} |\langle x, g_j\rangle|^2 + \dots$$

$$= \sum_{j=0}^{\infty} |\langle x, g_j\rangle|^2 \sum_{M=0}^{j} M^{2s-1}$$

$$\underbrace{\int_0^j y^{2s-1}\,dy}_{A j^{2s}} \le \sum_{M=0}^{j} M^{2s-1} \le \underbrace{\int_0^{j+1} y^{2s-1}\,dy}_{\le B j^{2s}}$$



verify that $\mathcal{E}_e[M] = o(M^{-2s})$

$$\mathcal{E}_e[M] \sum_{j=\lfloor M/2\rfloor}^{M-1} j^{2s-1} \le \sum_{j=\lfloor M/2\rfloor}^{M-1} j^{2s-1} \mathcal{E}_e[j] \le \sum_{j=\lfloor M/2\rfloor}^{\infty} j^{2s-1} \mathcal{E}_e[j]$$

$$\lim_{M\to\infty} \sum_{j=\lfloor M/2 \rfloor}^{\infty} j^{2s-1} \mathcal{E}_e[j] = 0$$

$$\sum_{j=\lfloor M/2 \rfloor}^{\infty} j^{2s-1} \geq C M^{2s}$$

$$\lim_{M\to\infty} C M^{2s} \mathcal{E}_e[M] = 0 \quad \Rightarrow \quad \mathcal{E}_e[M] = o(M^{-2s}) \text{ . q.e.d.}$$

$$W_{\mathcal{B},s} = \left\{ x \in \mathcal{H} : \sum_{j=0}^{\infty} j^{2s} |\langle x, g_j \rangle|^2 < \infty \right\}.$$

$\uparrow$

ONB

$W_{\mathcal{B},s}$ for $\mathcal{B}$ = Fourier or wavelet is a Sobolev space.

11.1.2 Regularity & Decay

Thm. 11.2. If $x \in L_1$, then $\hat{x}$ is uniformly continuous and satisfies

$$|\hat{x}(f)| \leq \int_{-\infty}^{\infty} |x(t)| dt < \infty, \quad f \in \mathbb{R}$$

and

$$\lim_{|f|\to\infty} \hat{x}(f) = 0.$$

$\hat{x} \in L_1 \Rightarrow x$ is unif. cont. & bounded and satisfies $\lim_{|t|\to\infty} x(t) = 0$.

$$x'(t) \circ\!\!-\!\!\bullet \ (2\pi i f) \hat{x}(f)$$

$$\hat{x}(f) = \int x(t) e^{-i2\pi f t} dt$$

Prop. 11.3. A function x is bounded and p times cont. diff.able with bounded derivative if

$$\int_{-\infty}^{\infty} |\hat{x}(f)| (1 + |f|)^p df < \infty.$$

Proof. $x^{(q)}(t) \circ\!\!-\!\!\bullet (2\pi i f)^q \hat{x}(f)$

$$|x^{(q)}(t)| \leq C \int_{-\infty}^{\infty} |\hat{x}(f)| |f|^q \leq C \int_{-\infty}^{\infty} |\hat{x}(f)| (1+|f|)^q df$$

$$\Rightarrow \int_{-\infty}^{\infty} |\hat{x}(f)| |f|^q df < \infty \quad , \text{for every } q \leq p .$$

$$\boxed{|\hat{x}(f)| \leq \frac{C'}{(1+|f|)^{p+1+\varepsilon}}}$$

11.1.3. Non-Linear Approximation

$\mathcal{B}$ ... ONB

$$x = \sum_{j=0}^{\infty} \langle x, g_j \rangle g_j \quad , \quad x_M = \sum_{j \in I_M} \langle x, g_j \rangle g_j$$

$$\varepsilon[M] = \|x - x_M\|^2 = \Big\| \sum_{j=0}^{\infty} \langle x, g_j \rangle g_j - \sum_{j \notin I_M} \langle x, g_j \rangle g_j \Big\|^2$$

$$= \Big\| \sum_{j \notin I_M} \langle x, g_j \rangle g_j \Big\|^2 = \sum_{j \notin I_M} |\langle x, g_j \rangle|^2$$

$$\|x\|^2 = \sum_j |\langle x, g_j \rangle|^2 = \sum_{j \in I_M} |\langle x, g_j \rangle|^2 + \sum_{j \notin I_M} |\langle x, g_j \rangle|^2$$

$$\varepsilon[M] = \|x\|^2 - \sum_{j \in I_M} |\langle x, g_j \rangle|^2$$