Lemma 1.34. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\widehat{g_k}\}_{k\in K}$ its canonical dual. Then, for each $m \in K$, we have

$$\sum_{k\neq m} |\langle g_m, \widehat{g_k}\rangle|^2 = \frac{1 - |\langle g_m, \widehat{g_m}\rangle| - |1 - \langle g_m, \widehat{g_m}\rangle|^2}{2}$$

Theorem 1.35. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\widehat{g_k}\}_{k\in K}$ its canonical dual. Then,

1. $\{g_k\}_{k\in K}$ is exact iff $\langle g_m, \widehat{g_m}\rangle = 1$, $\forall m \in K$
2. $\{g_k\}_{k\in K}$ is inexact iff there exists at least one $m \in K$ s.t. $\langle g_m, \widehat{g_m}\rangle \neq 1$.

Proof. $\langle g_m, \widehat{g_m}\rangle = 1$, $\forall m \in K$, $\Rightarrow$ $\{g_k\}_{\substack{k\neq m \\ k\in K}}$ is incomplete for $\mathcal{H}$ and hence $\{g_k\}$ is exact.

Fix $m \in K$, $\sum_{k\neq m} |\langle g_m, \widehat{g_k}\rangle|^2 = 0$

$$\Rightarrow \langle g_m, \tilde{g}_k \rangle = 0 \quad , \forall k, \text{except } k \neq m$$

$$\langle g_m, S^{-1} g_k \rangle = \langle S^{-1} g_m, g_k \rangle$$

$$= \langle \tilde{g}_m, g_k \rangle = 0$$

$$\tilde{g}_m \neq 0 \quad . \square$$

Corollary 1.36. Let $\{g_k\}_{k \in K}$ be a frame for $\mathcal{H}$. If $\{g_k\}_{k \in K}$ is exact, then $\{g_k\}_{k \in K}$ and its canonical dual $\{\tilde{g}_k\}_{k \in K}$ are biorthonormal, i.e.,

$$\langle g_m, \tilde{g}_k \rangle = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}.$$

Conversely, if $\{g_k\}_{k \in K}$ and $\{\tilde{g}_k\}_{k \in K}$ are biorthonormal, then $\{g_k\}_{k \in K}$ is exact.

Proof. $\{g_k\}_{k \in K}$ is exact $\overset{\text{Thm. 1.35}}{\Longrightarrow}$ $\langle g_m, \tilde{g}_m \rangle = 1, \forall m \in K$

$\overset{\text{Lem. 1.34}}{\Longrightarrow}$ $\langle g_m, \tilde{g}_k \rangle = 0, \forall k \neq m$

biorthonormality $\Rightarrow$ exact

$$\langle g_m, \tilde{g}_m \rangle = 1, \ \forall m \in \mathcal{K} \overset{\text{Thm. 1.35}}{\Longrightarrow} \{g_k\}_{k \in \mathcal{K}} \text{ is exact}$$

Thm. 1.37. If $\{g_k\}$ is an exact frame for $\mathcal{H}$ and $x = \sum_k c_k g_k$ with $x \in \mathcal{H}$, then the c_k are unique and given by

$$c_k = \langle x, \tilde{g}_k \rangle$$

where $\{\tilde{g}_k\}_{k \in \mathcal{K}}$ is the canonical dual of $\{g_k\}_{k \in \mathcal{K}}$.

Proof.

$$x = \sum_k \langle x, \tilde{g}_k \rangle g_k$$

$$x = \sum_k c_k g_k$$

$$\langle x, \tilde{g}_m \rangle = \left\langle \sum_k c_k g_k, \tilde{g}_m \right\rangle$$

$$= \sum_k c_k \underbrace{\langle g_k, \tilde{g}_m \rangle} = c_m$$

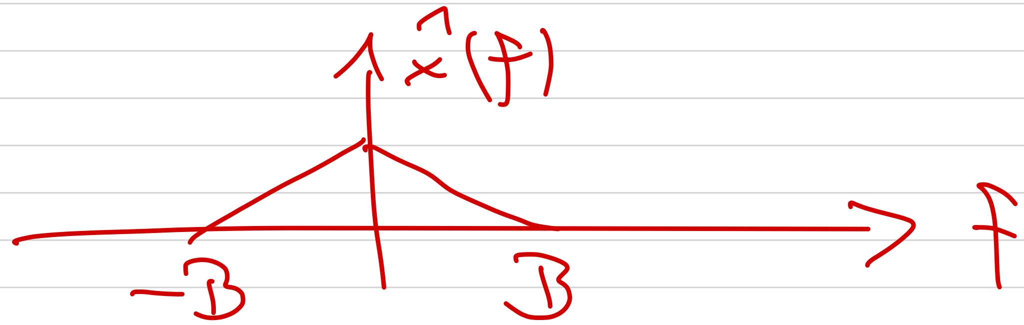
$$= \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}.$$

1.4. Sampling Theorem

$$x(t) \circ\!\!-\!\!\bullet\ \hat{x}(f) = \int x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi f \cdot} \rangle$$

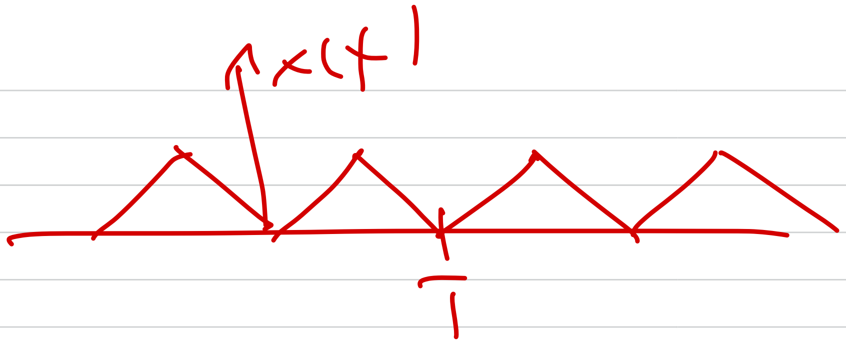
$$x(t) = \int \underbrace{\hat{x}(f)}_{\langle x(\cdot),\, e^{i2\pi f \cdot} \rangle} e^{i2\pi f t}\, df = \int \langle x(\cdot), e^{i2\pi f \cdot} \rangle\, e^{i2\pi f t}\, df$$

$x(t)$ is B-bandlimited if $\hat{x}(f) = 0$ for $|f| \geq B$



$$\underbrace{\sum_k x(t + kT)}_{y(t)} = \frac{1}{T} \sum_k \hat{x}\!\left(\frac{k}{T}\right) e^{i2\pi t \frac{k}{T}}$$

Poisson summation formula



$$y(t) = y(t+T) = \sum_{k} x(t+kT+T) = \sum_{k} x(t+(k+1)T)$$

$$k' = k+1$$

$$= \sum_{k'} x(t+k'T) = y(t)$$

$$c_l = \frac{1}{T}\int_0^T \sum_k x(t+kT)\, e^{-i2\pi l \frac{t}{T}}\, dt$$

$$= \frac{1}{T}\sum_k \int_0^T x(t+kT)\, e^{-i2\pi l \frac{t}{T}}\, dt =$$

$$t' = t+kT$$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t')\, e^{-i2\pi l \frac{t'-kT}{T}}\, dt'$$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t)\, e^{-i2\pi l \frac{t}{T}}\, dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, e^{-i 2\pi \ell \frac{t}{T}}\, dt = \frac{1}{T} \hat{x}\left(\frac{\ell}{T}\right)$$

$$\boxed{\hat{x}_d(f) = \sum_{k=-\infty}^{\infty} x(kT)\, e^{-i2\pi k f} \overset{\text{P.S.F.}}{=} \frac{1}{T} \sum_{\ell=-\infty}^{\infty} \hat{x}\left(\frac{f+\ell}{T}\right)}$$

(DTFT)

$$\frac{f}{T} = B \Rightarrow f = BT$$

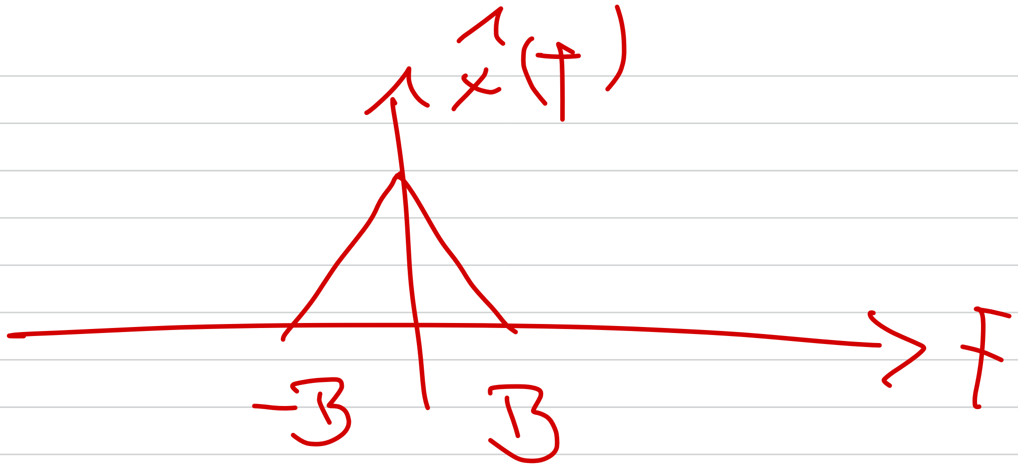
P.S.F. $\sum_k x(k) = \sum_k \hat{x}(k)$

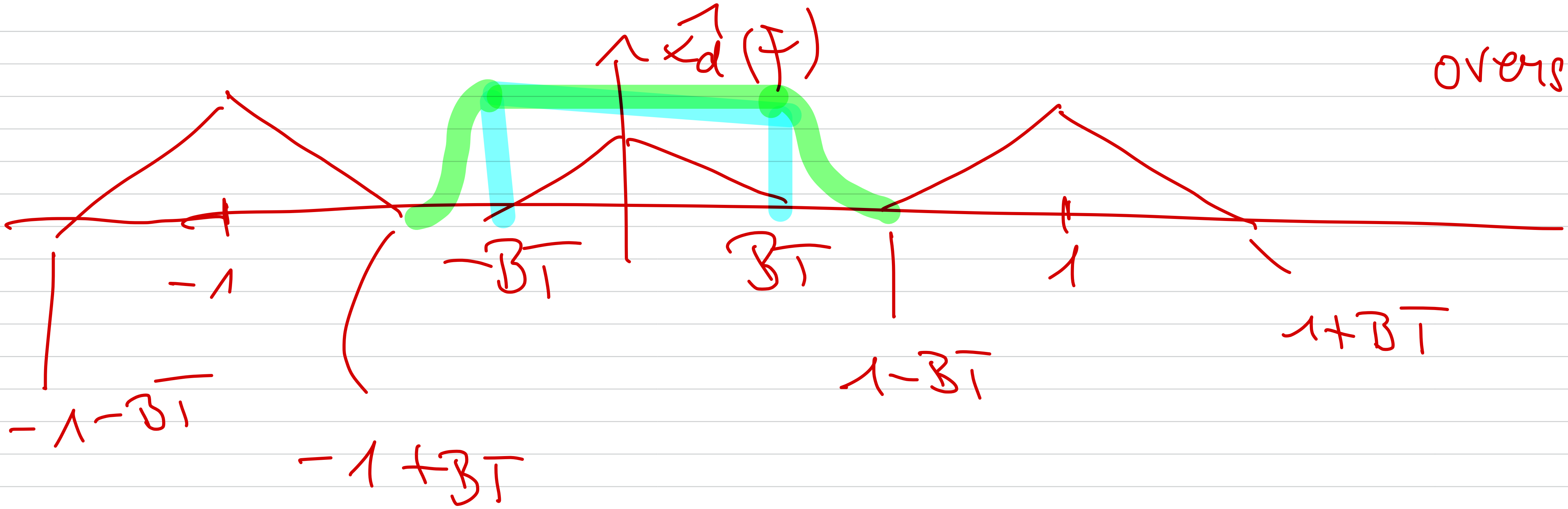
$\uparrow$

$x(t \cdot T)\, e^{-i2\pi t f}$

$\ell = -1$: $\frac{f-1}{T} = \pm B$

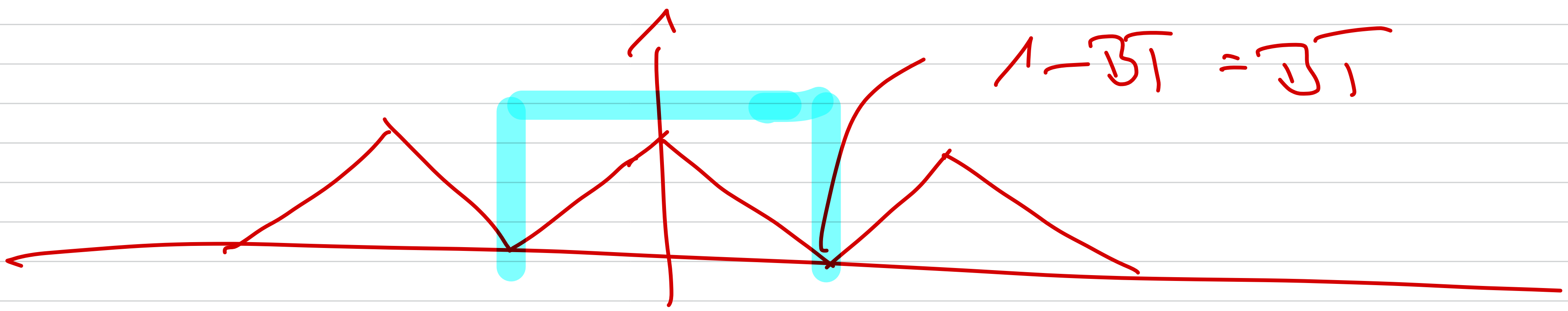
$f = 1 \pm BT$

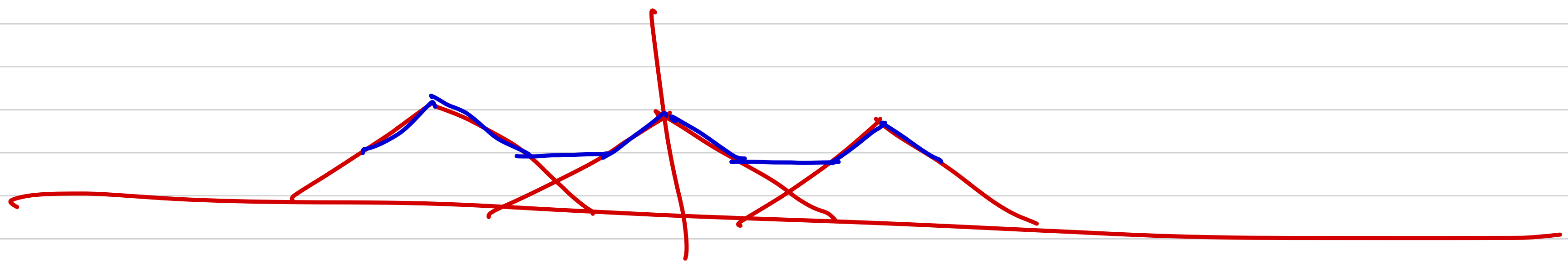




oversampled case

critical sampling





undersampling

$1 - BT < BT$

$$1 - BT \geq BT$$

$$1 \geq 2BT$$

$$\frac{1}{T} \geq 2B$$

$$\Uparrow$$

$f_s = \frac{1}{T}$ ... sampling rate

$$\hat{x}_d(f)\,T\,\hat{h}_{LP}(f) = \hat{x}(fT) \quad \Big| \quad f \to fT$$

$$\hat{x}(f) = \hat{x}_d(fT)\,T\,\hat{h}_{LP}(fT)$$

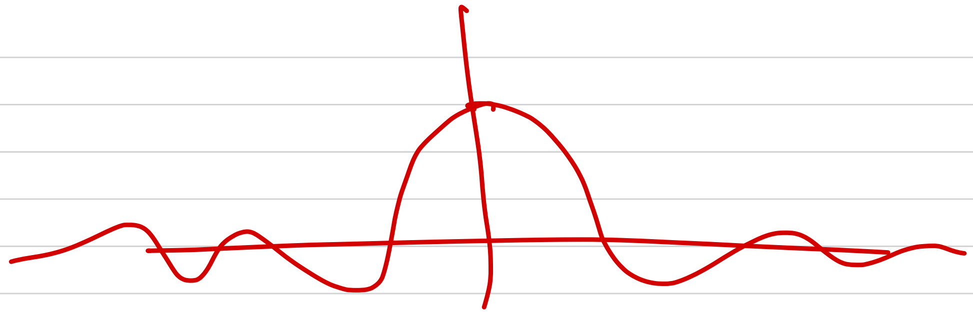
$$\hat{h}_{LP}(f) = \begin{cases} 1, & |f| \leq BT \\ 0, & \text{else} \end{cases}$$

$$\hat{x}(f) = \underbrace{T\,\hat{h}_{LP}(fT) \sum_{k} x(kT)\, e^{-i2\pi kf}}$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\, e^{i2\pi ft}\, df$$

$$= 2BT \sum_{k=-\infty}^{\infty} x(kT)\, \operatorname{sinc}(2B(t-kT))$$

$$\operatorname{sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}$$



Thm. 1.38. Sampling theorem. Let $x \in \mathcal{L}^2(B)$. Then, $x(t)$ is uniquely specified by its samples $x(kT)$, $k \in \mathbb{Z}$, if $\frac{1}{T} \geq 2B$. Specifically, we can reconstruct $x(t)$ according to

$$x(t) = 2BT \sum_k x(kT)\, \mathrm{sinc}(2B(t-kT)).$$

1.4.1. Sampling theorem as a frame expansion

$$g_k(t) = 2B\, \mathrm{sinc}(2B(t-kT))$$

$$x(kT) = \int_{-B}^{B} \hat{x}(f)\, e^{i2\pi kTf}\, df = \langle \hat{x}, \hat{g}_k \rangle = \langle x, g_k \rangle$$

$$\hat{g}_k(f) = \begin{cases} e^{-i2\pi kTf}, & |f| \leq B \\ 0, & \text{else} \end{cases}$$

$$x(t) = T \sum_k \langle x, g_k \rangle g_k(t)$$

with $g_k(t) = 2B \operatorname{sinc}(2B(t - kT))$

$$\|x\|^2 = \langle x, x \rangle = \left\langle T \sum_k \langle x, g_k \rangle g_k, x \right\rangle$$

$$= T \sum_k |\langle x, g_k \rangle|^2$$

$$\frac{1}{T} \|x\|^2 = \sum_k |\langle x, g_k \rangle|^2 = \langle Sx, x \rangle$$

$$\langle Sx, x \rangle = \frac{1}{T} \|x\|^2 \Rightarrow \text{tight with } A = \frac{1}{T}$$

$$T : x \Rightarrow \{\langle x, g_k \rangle\}_{k \in \mathbb{Z}}$$

$$T^* : \{c_k\}_{k \in \mathbb{Z}} \Rightarrow \sum_k c_k g_k$$

$$\tilde{g}_k = S^{-1} g_k = T g_k$$

$$S = \frac{1}{T} \mathbb{I}$$

$$\langle g_k, \tilde{g}_k \rangle = \langle g_k, T g_k \rangle = T \|g_k\|^2 = T \|\hat{g}_k\|^2 = 2BT$$

critical s.

$$f_s = \frac{1}{T} = 2B \Rightarrow 2BT = 1$$

crit. samp. $\langle g_k, \tilde{g}_k \rangle = 2BT = 1 \,, \forall k \in \mathbb{Z} \Rightarrow$ exact

$$2BT = \frac{2B}{\frac{1}{T}} = \frac{2B}{f_s}$$

$$g_k'(t) = \sqrt{T}\, g_k$$

$$x(t) = T \sum_k \langle x, g_k \rangle g_k = \sum_k \langle x, g_k' \rangle g_k'$$

$$\Rightarrow A = 1$$

$$\|g_k'\|^2 = T\|g_k\|^2 = T\|\hat{g}_k\|^2 = 2BT \overset{c.s.}{=} 1$$

$\Rightarrow$ ONB in the case of c.s.

$$\langle g_k, g_k \rangle = 2BT = \frac{2B}{f_s} \overset{u.s.}{\neq} 1 \Rightarrow \{g_k\}_{k\in\mathbb{Z}} \text{ is inexact}$$

Chapter 2. Uncertainty relations and sparse signal recovery

frequency extent

$\sigma_t \, \sigma_f \geq$ const.

time-duration

$$x(at) \circ\!\!-\!\!\bullet \frac{1}{|a|}\hat{x}(f/a)$$

$$x(t) = \int \hat{x}(f) e^{i2\pi ft} df$$

$$\hat{x}(f) = \int x(t) e^{-i2\pi ft} dt$$

$$x(t) = \int \langle x(\cdot), e^{2\pi i f \cdot} \rangle e^{i2\pi ft} df$$

$$x(t) = \int x(t') \delta(t-t') dt'$$

$$= \langle x(\cdot), \delta(t-\cdot) \rangle$$

$$x(t) =$$

$$x = \sum_i \langle x, e_i \rangle e_i$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \dots$$

$$x = \sum_i \langle x, f_i \rangle f_i$$

$$F = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots \\ \vdots & \omega^2 & \cdots & \\ & & & \\ 1 & ? & & \end{bmatrix}$$

$$\omega = e^{-i2\pi/m}$$

$$f_1 \quad f_2 \quad \dots$$

$$F F^H = F^H F = I_m$$

Notation. U -- unitary

$$P_{\mathcal{A}}(U) = U D_{\mathcal{A}} U^H$$

$$D_{\mathcal{A}} = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \qquad \mathcal{A} = \{1, 3, 7, \dots\}$$

$$P_{\mathcal{A}}(U) = \sum_{i \in \mathcal{A}} u_i u_i^H$$

$$W^{U,\mathcal{A}} = R(P_{\mathcal{A}}(U))$$

$$x_{\mathcal{A}} = D_{\mathcal{A}} x$$

$$|||A|||_2 = \max_{x : \|x\|_2 = 1} \|Ax\|_2 \qquad \text{op. 2-norm}$$