Lemma 1.34. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\widehat{g_k}\}_{k\in K}$ its canonical dual. Then, for each $m\in K$, we have

$$\sum_{k\neq m} |\langle g_m, \widehat{g_k}\rangle|^2 = \frac{1-|\langle g_m,\widehat{g_m}\rangle| - |1-\langle g_m,\widehat{g_m}\rangle|^2}{2}$$

Theorem 1.35. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\widehat{g_k}\}_{k\in K}$ its canonical dual. Then,

1. $\{g_k\}_{k\in K}$ is exact iff $\langle g_m,\widehat{g_m}\rangle = 1$, $\forall m\in K$
2. $\{g_k\}_{k\in K}$ is inexact iff there exists at least one $m\in K$ s.t. $\langle g_m,\widehat{g_m}\rangle \neq 1$.

Proof. $\langle g_m,\widehat{g_m}\rangle = 1$, $\forall m\in K$, $\Rightarrow$ $\{g_k\}_{\substack{k\neq m\\ k\in K}}$ is incomplete for $\mathcal{H}$ and hence $\{g_k\}$ is exact.

Fix $m\in K$, $\sum_{k\neq m} |\langle g_m,\widehat{g_k}\rangle|^2 = 0$

$$\Rightarrow \quad \langle g_m, \tilde{g_k} \rangle = 0 \quad , \forall k, \text{except } k \neq m$$

$$\langle g_m, S^{-1} g_k \rangle = \langle S^{-1} g_m, g_k \rangle$$

$$= \langle \tilde{g_m}, g_k \rangle = 0$$

$$\tilde{g_m} \neq 0 \quad . \square$$

Corollary 1.36. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$. If $\{g_k\}_{k\in K}$ is exact, then $\{g_k\}_{k\in K}$ and its canonical dual $\{\tilde{g_k}\}_{k\in K}$ are biorthonormal, i.e.,

$$\langle g_m, \tilde{g_k} \rangle = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}.$$

Conversely, if $\{g_k\}_{k\in K}$ and $\{\tilde{g_k}\}_{k\in K}$ are biorthonormal, then $\{g_k\}_{k\in K}$ is exact.

Proof. $\{g_k\}_{k\in K}$ is exact $\overset{\text{Thm. 1.35}}{\Longrightarrow}$ $\langle g_m, \tilde{g_m} \rangle = 1, \forall m \in K$

$\overset{\text{Lem. 1.34}}{\Longrightarrow}$ $\langle g_m, \tilde{g_k} \rangle = 0, \forall k \neq m$

biorthonormality $\Rightarrow$ exact

$$\langle g_m, \tilde{g}_m \rangle = 1, \ \forall m \in \mathbb{K} \overset{\text{Thm. 1.35}}{\Longrightarrow} \{g_k\}_{k \in \mathbb{K}} \text{ is exact}$$

Thm. 1.37. If $\{g_k\}$ is an exact frame for $\mathcal{H}$ and $x = \sum_k c_k g_k$ with $x \in \mathcal{H}$, then the c_k are unique and given by

$$c_k = \langle x, \tilde{g}_k \rangle$$

where $\{\tilde{g}_k\}_{k \in \mathbb{K}}$ is the canonical dual of $\{g_k\}_{k \in \mathbb{K}}$.

Proof.

$$x = \sum_k \langle x, \tilde{g}_k \rangle g_k$$

$$x = \sum_k c_k g_k$$

$$\langle x, \tilde{g}_m \rangle = \left\langle \sum_k c_k g_k, \tilde{g}_m \right\rangle$$

$$= \sum_k c_k \underbrace{\langle g_k, \tilde{g}_m \rangle} = c_m$$

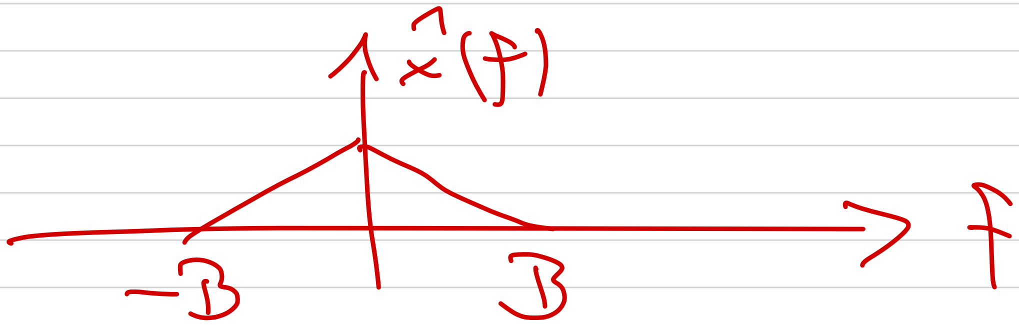
$$= \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}.$$

1.4. Sampling Theorem

$$x(t) \circ\!\!-\!\!\bullet\ \hat{x}(f) = \int x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi f \cdot} \rangle$$

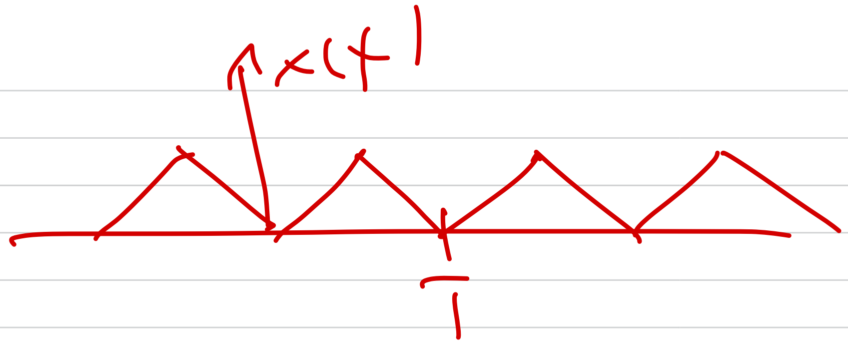
$$x(t) = \int \underbrace{\hat{x}(f)}_{\langle x(\cdot),\, e^{i2\pi f \cdot} \rangle} e^{i2\pi f t}\, df = \int \langle x(\cdot), e^{i2\pi f \cdot} \rangle\, e^{i2\pi f t}\, df$$

$x(t)$ is B-bandlimited if $\hat{x}(f) = 0$ for $|f| \geq B$



$$\underbrace{\sum_k x(t + kT)}_{y(t)} = \frac{1}{T} \sum_k \hat{x}\left(\frac{k}{T}\right) e^{i2\pi t \frac{k}{T}}$$

Poisson summation formula



$$y(t) = y(t+T) = \sum_{k} x(t+kT+T) = \sum_{k} x(t+(k+1)T)$$

$k' = k+1$

$$= \sum_{k'} x(t+k'T) = y(t)$$

$$c_\ell = \frac{1}{T}\int_0^T \sum_k x(t+kT)\, e^{-i2\pi\ell\frac{t}{T}}\, dt$$

$$= \frac{1}{T}\sum_k \int_0^T x(t+kT)\, e^{-i2\pi\ell\frac{t}{T}}\, dt =$$

$t' = t+kT$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t')\, e^{-i2\pi\ell\frac{t'-kT}{T}}\, dt'$$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t)\, e^{-i2\pi\ell\frac{t}{T}}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-i2\pi \ell \frac{t}{T}}\, dt = \frac{1}{T}\hat{x}\left(\frac{\ell}{T}\right)$$

$$\boxed{\begin{aligned} \hat{x}_d(f) &= \sum_{k=-\infty}^{\infty} x(kT)\, e^{-i2\pi k f} \\ &\overset{\text{P.S.F.}}{=} \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{x}\left(\frac{f+k}{T}\right) \end{aligned}}$$

(DTFT)

$$\frac{f}{T} = \beta \Rightarrow f = \beta T$$

P.S.F.

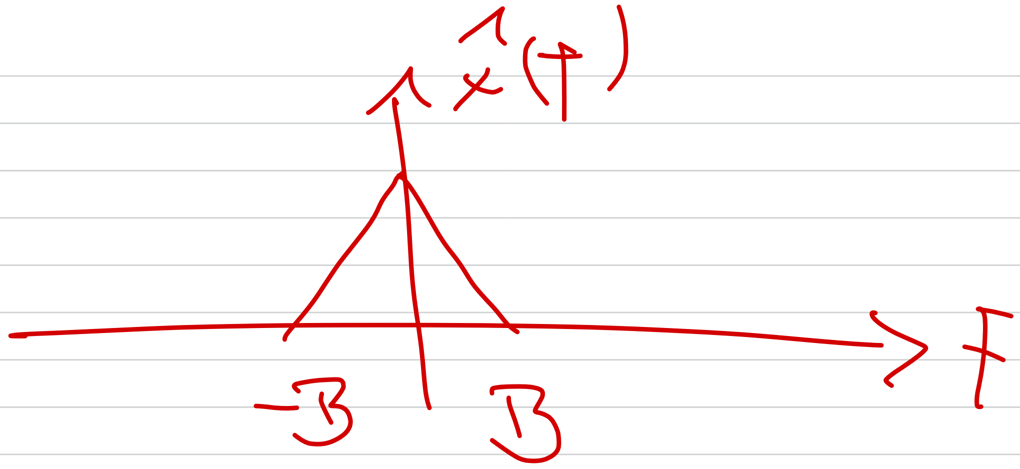
$$\sum_k x(k) = \sum_k \hat{x}(k)$$

$$\uparrow$$

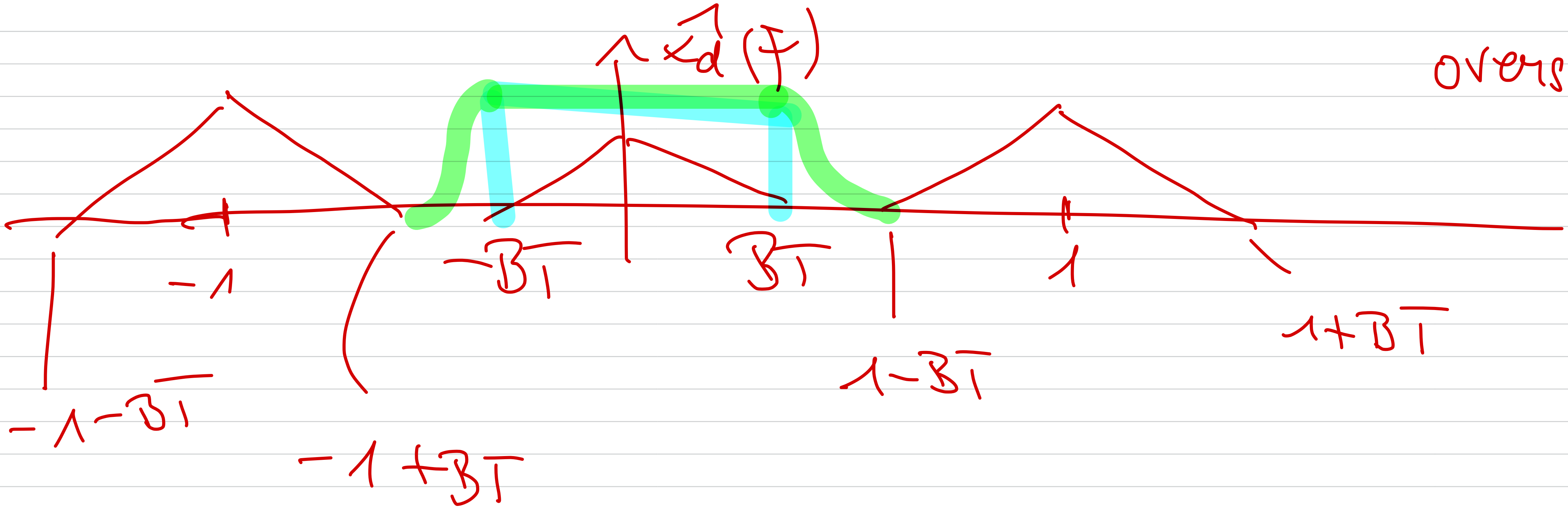
$$x(t \cdot T)\, e^{-i2\pi t f}$$

$$k=-1: \quad \frac{f-1}{T} = \pm\beta$$

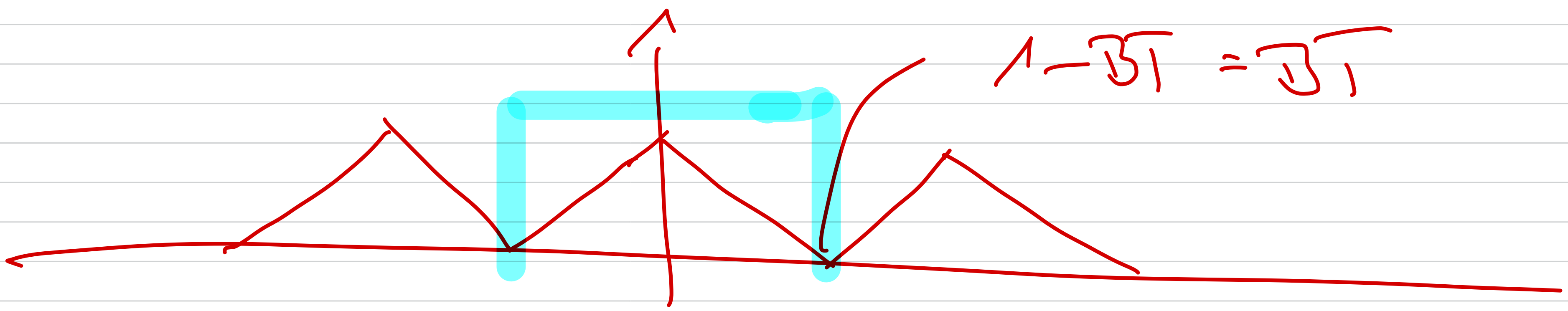
$$f = 1 \pm \beta T$$

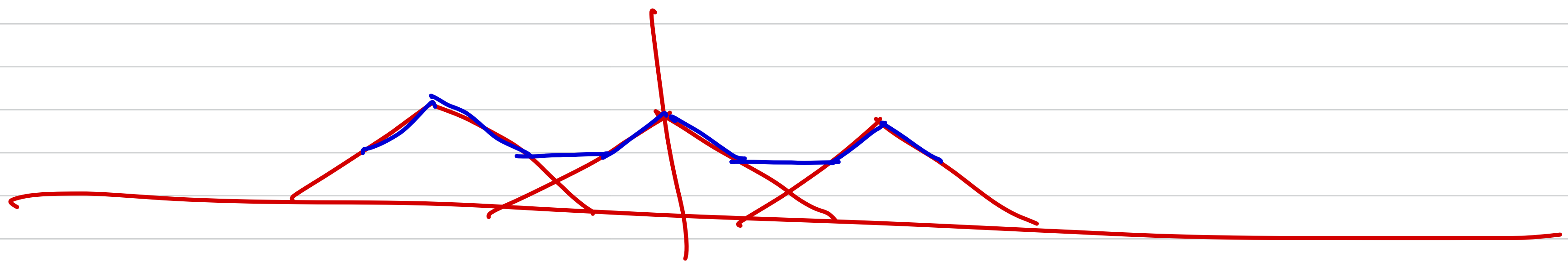


oversampled case



critical sampling





undersampling

$$1 - B_T < B_T$$

$$1 - B_T \geq B_T$$

$$1 \geq 2B_T$$

$$\frac{1}{T} \geq 2B$$

$\uparrow$

$f_s = \frac{1}{T}$ ... sampling rate

$$\hat{x}_d(f)\,T\,\hat{h}_{LP}(f) = \hat{x}(fT) \quad \Big| \quad f \to fT$$

$$\hat{x}(f) = \hat{x}_d(fT)\,T\,\hat{h}_{LP}(fT)$$

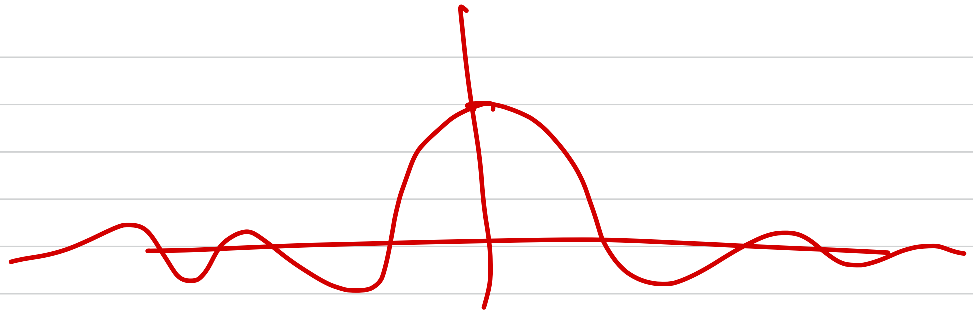
$$\hat{h}_{LP}(f) = \begin{cases} 1, & |f| \leq BT \\ 0, & \text{else} \end{cases}$$

$$\hat{x}(f) = \underbrace{T\,\hat{h}_{LP}(fT)\sum_{k} x(kT)\,e^{-i2\pi kf}}$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\,e^{i2\pi ft}\,df$$

$$= 2BT \sum_{k=-\infty}^{\infty} x(kT)\,\operatorname{sinc}(2B(t-kT))$$

$$\operatorname{sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}$$



Thm. 1.38. Sampling theorem. Let $x \in \mathcal{L}^2(B)$. Then, $x(t)$ is uniquely specified by its samples $x(kT)$, $k \in \mathbb{Z}$, if $\frac{1}{T} \geq 2B$. Specifically, we can reconstruct $x(t)$ according to

$$x(t) = 2BT \sum_k x(kT)\, \mathrm{sinc}(2B(t-kT)).$$

1.4.1. Sampling theorem as a frame expansion

$$g_k(t) = 2B\, \mathrm{sinc}(2B(t-kT))$$

$$x(kT) = \int_{-B}^{B} \hat{x}(f)\, e^{i2\pi kTf}\, df = \langle \hat{x}, \hat{g}_k \rangle = \langle x, g_k \rangle$$

$$\hat{g}_k(f) = \begin{cases} e^{-i2\pi kTf}, & |f| \leq B \\ 0, & \text{else} \end{cases}$$

$$x(t) = T \sum_{k} \langle x, g_k \rangle g_k(t)$$

with $g_k(t) = 2B \operatorname{sinc}(2B(t - kT))$

$$\|x\|^2 = \langle x, x \rangle = \left\langle T \sum_{k} \langle x, g_k \rangle g_k, x \right\rangle$$

$$= T \sum_{k} |\langle x, g_k \rangle|^2$$

$$\frac{1}{T} \|x\|^2 = \sum_{k} |\langle x, g_k \rangle|^2 = \langle Sx, x \rangle$$

$\langle Sx, x \rangle = \frac{1}{T} \|x\|^2 \Rightarrow$ tight with $A = \frac{1}{T}$

$$T : x \Rightarrow \{ \langle x, g_k \rangle \}_{k \in \mathbb{Z}}$$

$$T^* : \{c_k\}_{k \in \mathbb{Z}} \Rightarrow \sum_{k} c_k g_k$$

$$\tilde{g_k} = S^{-1} g_k = T g_k$$

$$S = \frac{1}{T} \mathbb{I}$$

$$\langle g_k, \tilde{g_k} \rangle = \langle g_k, T g_k \rangle = T \|g_k\|^2 = T \|\hat{g_k}\|^2 = 2BT$$

critical s.

$$f_s = \frac{1}{T} = 2B \Rightarrow 2BT = 1$$

crit. samp. $\langle g_k, \tilde{g_k} \rangle = 2BT = 1, \ \forall k \in \mathbb{Z} \Rightarrow$ exact

$$2BT = \frac{2B}{\frac{1}{T}} = \frac{2B}{f_s}$$

$$g_k'(t) = \sqrt{T}\, g_k$$

$$x(t) = T \sum_k \langle x, g_k \rangle g_k = \sum_k \langle x, g_k' \rangle g_k'$$

$$\Rightarrow A = 1$$

$$\|g_k'\|^2 = T\|g_k\|^2 = T\|\hat{g}_k\|^2 = 2BT \overset{c.s.}{=} 1$$

$\Rightarrow$ ONB in the case of c.s.

$$\langle g_k, g_k \rangle = 2BT = \frac{2B}{f_S} \overset{u.s.}{\neq} 1 \Rightarrow \{g_k\}_{k\in\mathbb{Z}} \text{ is inexact}$$

Chapter 2. Uncertainty relations and sparse signal recovery

frequency extent

$$\sigma_t \, \sigma_f \geq \text{const.}$$

time-duration

$$x(t) = \int \hat{x}(f) e^{i2\pi ft} df$$

$$\hat{x}(f) = \int x(t) e^{-i2\pi ft} dt$$

$$x(at) \circ\!\!-\!\!\bullet \frac{1}{|a|} \hat{x}(f/a)$$

$$x(t) = \int \langle x(\cdot), e^{2\pi i f \cdot} \rangle e^{i2\pi ft} df$$

$$x(t) = \int x(t')\,\delta(t-t')\,dt'$$

$$= \langle x(\cdot), \delta(t-\cdot) \rangle$$

$x(t) =$

$$x = \sum_i \langle x, e_i \rangle e_i$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \ldots$$

$$x = \sum_i \langle x, f_i \rangle f_i$$

$$F = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & \text{—} & 1 \\ 1 & \omega & \omega^2 & \ldots \\ \vdots & \omega^2 & \ldots & \\ 1 & ? & & \end{bmatrix}$$

$$\omega = e^{-i2\pi/m}$$

$f_1 \quad f_2 \quad \ldots$

$$F F^H = F^H F = I_m$$

Notation. U -- unitary

$$P_{\mathcal{A}}(U) = U D_{\mathcal{A}} U^H \qquad \mathcal{A} = \{1, 3, 7, \dots\}$$

$$D_{\mathcal{A}} = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}, \quad D_{\mathcal{A}} x = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ \vdots \\ x_7 \end{pmatrix}$$

$$P_{\mathcal{A}}(U) = \sum_{i \in \mathcal{A}} u_i u_i^H, \quad U = F \text{ DFT matrix}$$

$$W^{U,\mathcal{A}} = R(P_{\mathcal{A}}(U))$$

$$x_{\mathcal{A}} = D_{\mathcal{A}} x$$

$$(Ax)^H Ax = x^H A^H A x$$

$$|||A|||_2 = \max_{x : \|x\|_2 = 1} \|Ax\|_2 \quad \text{op. 2-norm}$$

$$\|A\|_2 = \sqrt{\operatorname{tr}(A A^H)} = \sqrt{\sum_{i,j} |A_{ij}|^2}$$

$$\Delta_{P,Q}(U) = |||D_P\, P_Q(U)|||_2$$

e.g.f.

Lemma 2.20.

$$\Delta_{P,Q}(U) = \max_{x \in W^{u,Q} \setminus \{0\}} \frac{\|x_P\|_2}{\|x\|_2}$$

$$D_P = I D_P I^H, \quad P_P(A) = A D_P A^H \quad AA^H = I$$

$$P_Q(U) = U D_Q U^H, \quad P_Q(B) = B D_Q B^H \quad BB^H = I$$

$$|||P_P(A) P_Q(B)|||_2 = |||A D_P A^H B D_Q B^H|||_2$$

$$= |||A^H A D_P A^H B D_Q B^H|||_2$$

$$= |||D_P \underbrace{A^H B}_{U} D_Q \underbrace{B^H}_{U^H} A|||_2 \quad x: \|x\|_2 = 1 \quad Ax$$

$$= |||D_P U D_Q U^H|||_2 = |||D_P P_Q(U)|||_2$$

An uncertainty relation is an inequality of the form

$$\Delta_{P,Q}(U) \leq c < 1.$$

In principle the norm $|||D_P P_Q(U)|||_2$ is difficult to quantify

later: In the case $U = F$, we can get an explicit expression for $|||D_P P_Q(U)|||_2$, but this is an absolute exception

$$\frac{|||D_P P_Q(U)|||_2}{\sqrt{\operatorname{rank}(D_P P_Q(U))}} \leq \Delta_{P,Q}(U) \leq |||D_P P_Q(U)|||_2 = \sqrt{\operatorname{tr}(D_P \underbrace{P_Q(U) P_Q(U)}_{P_Q(U)} D_P)} = \sqrt{\operatorname{tr}(D_P P_Q(U))}$$

$$\operatorname{rank}(D_P P_Q(U)) = \operatorname{rank}(\underline{D_P}\ \underline{U D_Q U^H}) \leq \min(|P|, |Q|)$$

$$\sqrt{\frac{\operatorname{tr}(D_P P_Q(U))}{\min(|P|,|Q|)}} \leq \Delta_{P,Q}(U) \leq \sqrt{\operatorname{tr}(D_P P_Q(U))}$$

$F \in \mathbb{C}^{m \times m}$

$[F]_{k,l} = F_{k,l} = \frac{1}{\sqrt{m}} e^{i \frac{k l 2\pi}{m}}$

particularize to $U = F$, and get

$$\sqrt{\operatorname{tr}(D_P P_Q(F))} = \sqrt{\operatorname{tr}(D_P F D_Q F^H)} = \sqrt{\sum_{i \in P} \sum_{j \in Q} |F_{i,j}|^2} = \sqrt{\frac{|P||Q|}{m}}$$

$$\sqrt{\frac{\max(|P|,|Q|)}{m}} \leq \Delta_{P,Q}(F) \leq \sqrt{\frac{|P||Q|}{m}}$$

Ex. 1. $P = \{1\}$, $Q = \{1, \dots, m\}$

$$1 \leq \Delta_{P,Q}(F) \leq 1 \Rightarrow \Delta_{P,Q}(F) = 1$$

Ex. 2 $P = \left\{ \frac{m}{n}, \frac{2m}{n}, \dots, \frac{(n-1)m}{n}, m \right\}$, n div. m

$Q = \{ l+1, \dots, l+n \}$

$l \in \{1, \dots, m\}$

$$\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(F) \leq \sqrt{\frac{n^2}{m}} = \frac{n}{\sqrt{m}}$$

$$\sqrt{\frac{n}{m}} \leq \Delta_{P,Q}(F) \leq \frac{n}{\sqrt{m}} = \sqrt{n}\underbrace{\sqrt{\frac{n}{m}}}_{\text{const.}}$$

Lemma 2.1. Let n divide m and consider

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \right\}$$

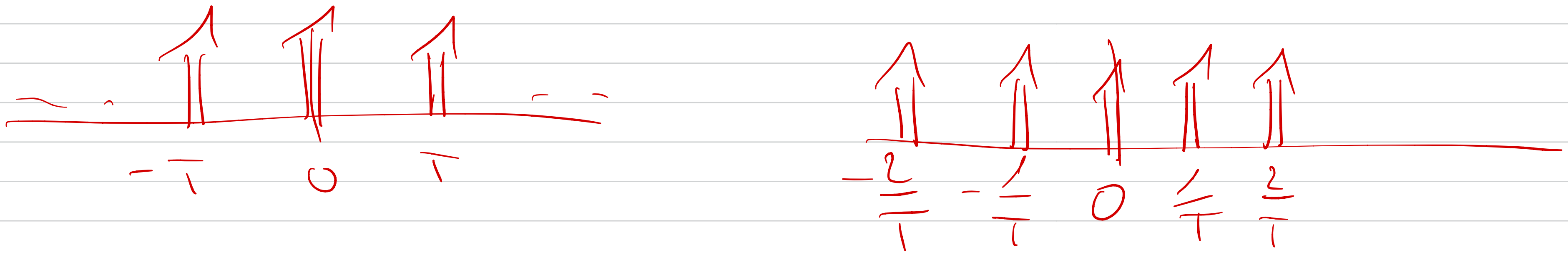
$$Q = \{ l+1, \ldots, l+n \}, \quad Q \text{ interpreted circularly}$$

Then, $\Delta_{P,Q}(F) = \sqrt{n/m}$.

Proof uses a discrete version of the following:

proof is highly specific to the Fourier transform

$$\sum_{l} \delta(t - lT) \;\circ\!\!-\!\!\bullet\; \frac{1}{T} \sum_{l} \delta\left(f - \frac{l}{T}\right)$$



Large-sieve:

$$\Delta_{P,Q}(F) \le \sqrt{2}\,\sqrt{n/m}$$

$$\left(\sqrt{\frac{n}{m}} \le \Delta_{P,Q}(F) \le \sqrt{n}\,\sqrt{\frac{n}{m}} \right)$$

2.2.2. Coherence-based uncertainty relations

Def. 2.3. For $A=(a_1,\dots,a_n)\in\mathbb{C}^{m\times n}$ with columns $\|\cdot\|_2$-normalized to 1, the coherence of A is defined as

$$\mu(A) := \max_{i\neq j} |a_i^H a_j| .$$

Lemma 2.4. Let $U\in\mathbb{C}^{m\times m}$ be unitary and $P,Q\subseteq\{1,\dots,m\}$. Then,

$$\Delta_{P,Q}(U) \le \sqrt{|P||Q|}\,\mu([I\; U]).$$

Proof.

$$\Delta_{P,Q}^2(U) \le \operatorname{tr}(D_P\, U D_Q U^H)$$

$$\left(\operatorname{tr}(D_P\, \underbrace{P_Q(U) P_Q(U)}_{P_Q(U)} D_P)\right)$$

$$= \sum_{q\in P} \sum_{\ell\in Q} |U_{q,\ell}|^2$$

$$\le |P|\,|Q| \max_{q,\ell} |U_{q,\ell}|^2$$

$$= |P||Q| \mu^2 (\mathcal{F}[I\ U]). \quad \square$$

2.2.3. Concentration inequalities

$$(s :=) p = \overline{F} q$$

p ← freq.-domain signal; q ← time-domain signal; $\overline{F}$ ↑ DFT

$$p = \overline{F} q$$

$$s := \overline{I} p = \overline{F} q$$

$$\overline{I} \Rightarrow A$$

$$\overline{F} \Rightarrow B$$

$$s := Ap = Bq$$

$$Ap = Bq \mid A^H.$$

$$p = \underbrace{A^H B}_{U} q = Uq$$

$$(s :=) \boxed{p = Uq}, \quad U \text{ unitary}$$

Def. 2.5. Let $P \subseteq \{1, \dots, m\}$ and $\varepsilon_P \in [0, 1]$. The vector $x \in \mathbb{C}^m$ is said to be ε_P - concentrated if $\|x - x_P\|_2 \le \varepsilon_P \|x\|_2$.

$$\frac{\|x-x_P\|_2}{\|x\|_2} \le \varepsilon_P$$

$\varepsilon_P = 0 \Rightarrow \dfrac{\|x-x_P\|_2}{\|x\|_2} = 0 \Rightarrow x = x_P \Rightarrow x$ is strictly supported on P

Lemma 2.6. Let $U \in \mathbb{C}^{m\times m}$ be unitary and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exists a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $p = Uq$. Then,

$$\Delta_{P,Q}(U) \ge [1-\varepsilon_P-\varepsilon_Q]_+ .$$

Proof. We have $\qquad p - P_Q(U)p + P_Q(U)p - P_Q(U)p_P$

$$\begin{aligned}
\|p - P_Q(U)p_P\|_2 &\le \|p - P_Q(U)p\|_2 + \|P_Q(U)p_P - P_Q(U)p\|_2 \\
&\le \|p - P_Q(U)p\|_2 + |||P_Q(U)|||_2 \, \|p_P - p\|_2 \\
&\le \|p - P_Q(U)p\|_2 + 1 \cdot \|p_P - p\|_2 \\
&= \|\overset{U^H}{p} - \overset{U^H}{U D_Q} \overset{q}{U^H p}\|_2 + \|p_P - p\|_2
\end{aligned}$$

$$(q = U^H p)$$
$$= \|q - q_Q\|_2 + \|p - p_P\|_2$$
$$\overset{\text{Def. 2.5.}}{\le} \varepsilon_Q \|q\|_2 + \varepsilon_P \|p\|_2 \qquad (= U^H p)$$
$$= (\varepsilon_P + \varepsilon_Q)\|p\|_2 .$$

$$P_Q(U) p_P = p - p + P_Q(U) p$$

$$\|P_Q(U) p_P\|_2 \overset{\text{rev. } \Delta\text{-inequ.}}{\ge} [\|p\|_2 - \underbrace{\|p - P_Q(U) p_P\|_2]_+}_{\le (\varepsilon_P + \varepsilon_Q)\|p\|_2}$$

$$\ge \|p\|_2 [1 - (\varepsilon_P + \varepsilon_Q)]_+ \quad \Big| \cdot \frac{1}{\|p\|_2}$$

$$\left\| P_Q(U) D_P \frac{p}{\|p\|_2} \right\|_2 \ge [1 - \varepsilon_P - \varepsilon_Q]_+ .$$

$$\Delta_{P,Q}(U) \ge [1 - \varepsilon_P - \varepsilon_Q]_+ . \qquad \square$$

Corollary 2.7. Let $A, B \in \mathbb{C}^{m \times m}$ and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exist a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $Ap = Bq$. Then,

$$|P||Q| \geq \frac{[1-\varepsilon_P-\varepsilon_Q]_+^2}{\mu^2([A\ B])}.$$

Proof. Let $U = A^H B$. Then,

$$[1-\varepsilon_P-\varepsilon_Q]_+ \overset{L2.6.}{\leq} \Delta_{P,Q}(U) \overset{L2.4.}{\leq} \sqrt{|P||Q|}\,\mu([I\ U])$$

$$\mu([I\ U]) = \mu([I\ A^H B]) = \mu([A\ B]). \ \square$$

1.st special case: $\varepsilon_P = \varepsilon_Q = 0 \Rightarrow$ Elad–Bruckstein result, 2003

Corollary 2.8. Let $A, B \in \mathbb{C}^{m\times m}$ be unitary. If $Ap = Bq$ for nonzero $p, q \in \mathbb{C}^m$, then

$$\|p\|_0 \|q\|_0 \geq \frac{1}{\mu^2([A\ B])}.$$

2.nd special case: $\overset{A=I}{U = F} \Rightarrow \mu([I\ F]) = \frac{1}{\sqrt{m}}$

$$\|p\|_0 \|q\|_0 \geq m. \quad \text{Donoho–Stark, 1989}$$

$$p = Uq = Fq$$

2.2.4. Noisy recovery in $(\mathbb{C}^m, \|\cdot\|_2)$

$p, n \in \mathbb{C}^m$

$P \subseteq \{1, \ldots, m\}$

$P^c = \{1, \ldots, m\} \setminus P$

observe

$$y = p_{P^c} + n$$

The samples indexed by P are lost in the observation and noise gets added

Q: Can we recover p from y?

In general, no!

However, if p satisfies certain structural properties and $|P|$ is not too large, then this is, indeed, possible.

sparsity

Lemma 2.9. Let $U \in \mathbb{C}^{m\times m}$ be unitary, $Q \subseteq \{1,\dots,m\}$, $p \in \mathcal{D}^{U,Q}$, and consider

(discard the elements of p supported on P)

$$y = P_Q p + n, \quad \text{add noise}$$

where $n \in \mathbb{C}^m$ and $P^c = \{1,\dots,m\}\setminus P$ with $P \subseteq \{1,\dots,m\}$. If

$$\left(\Delta_{P,Q}(U) = |||D_P P_Q(U)|||_2\right) \; \Delta_{P,Q}(U) < 1, \quad \text{recovery condition}$$

then there exists a matrix $L \in \mathbb{C}^{m\times m}$ s.t.

$$\|Ly - p\|_2 \le C\|\overbrace{n_{P^c}}^{\text{all the noise we see}}\|_2$$

with $C = \dfrac{1}{(1-\Delta_{P,Q}(U))}$.

In particular,

$$|P|\,|Q| < \frac{1}{\mu^2(C\,U)}$$

is sufficient for $\Delta_{P,Q}(U) < 1$. follows by application of L.2.4.

Proof. For $\Delta_{P,Q}(U) < 1$, it follows that $(I - D_P P_Q(U))$ is inv. with

$$|||(I - D_P P_Q(U))^{-1}|||_2 \le \frac{1}{1 - |||D_P P_Q(U)|||_2} = \frac{1}{1-\Delta_{P,Q}(U)}$$

(Neumann series expansion)

$$L = (I - D_p P_\Omega(u))^{-1} D_{pc}$$

$$L\, p_{pc} = (I - D_p P_\Omega(u))^{-1} \underbrace{D_{pc}\, p}_{p_{pc}}$$

$$p \in W^{u,0}$$

$$= (I - D_p P_\Omega(u))^{-1} (I - D_p) p$$

$$= (I - D_p P_\Omega(u))^{-1} (I - D_p P_\Omega(u)) p$$

$$= p$$

$$\| Ly - p \|_2 = \| L(p_{pc} + n) - p \|_2$$

$$= \| Ln \|_2$$

$$\leq \||(I - D_p P_\Omega(u))^{-1}\||_2 \, \| n_{pc} \|_2$$

$$\leq \frac{1}{1 - \Delta_{p,\Omega}(u)} \| n_{pc} \|_2 \quad \square$$

Note: In the noise-free case, $\Delta_{P,Q}(U) < 1 \Rightarrow$ perfect recovery.

$P = \{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \}$ $\leftarrow$ indicates what we throw away

$Q = \{ l+1, \ldots, l+n \}$ $\leftarrow$ carries p

$U = F$

p is n-sparse in $U = F$

we are missing n entries in the noisy observation y

$\Delta_{P,Q}(F) = \sqrt{n/m}$ $\quad (L.?.1)$

Stable recovery of p is possible for $n \leq m/2$

$m \geq 2n$

$\Delta_{P,Q}(F) \leq \frac{n}{\sqrt{m}} < 1$

$n < \sqrt{m}$

$m > n^2 \Rightarrow$ square-root bottleneck