Lemma 1.34. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\tilde{g}_k\}_{k\in K}$ its canonical dual. Then, for each $m\in K$, we have

$$\sum_{k\neq m} |\langle g_m, \tilde{g}_k\rangle|^2 = \frac{1-|\langle g_m,\tilde{g}_m\rangle| - |1-\langle g_m,\tilde{g}_m\rangle|^2}{2}$$

Theorem 1.35. Let $\{g_k\}_{k\in K}$ be a frame for $\mathcal{H}$ and $\{\tilde{g}_k\}_{k\in K}$ its canonical dual. Then,

1. $\{g_k\}_{k\in K}$ is exact iff $\langle g_m, \tilde{g}_m\rangle = 1$, $\forall m\in K$
2. $\{g_k\}_{k\in K}$ is inexact iff there exists at least one $m\in K$ s.t. $\langle g_m, \tilde{g}_m\rangle \neq 1$.

Proof. $\langle g_m, \tilde{g}_m\rangle = 1$, $\forall m\in K$, $\Rightarrow$ $\{g_k\}_{\substack{k\neq m\\ k\in K}}$ is incomplete for $\mathcal{H}$ and hence $\{g_k\}$ is exact.

Fix $m\in K$, $\sum_{k\neq m} |\langle g_m, \tilde{g}_k\rangle|^2 = 0$

$$\Rightarrow \langle g_m, \tilde{g_k} \rangle = 0 \quad , \forall k, \text{except } k \neq m$$

$$\langle g_m, S^{-1} g_k \rangle = \langle S^{-1} g_m, g_k \rangle$$

$$= \langle \tilde{g_m}, g_k \rangle = 0$$

$$\tilde{g_m} \neq 0 \quad . \square$$

Corollary 1.36. Let $\{g_k\}_{k\in\mathbb{K}}$ be a frame for $\mathcal{H}$. If $\{g_k\}_{k\in\mathbb{K}}$ is exact, then $\{g_k\}_{k\in\mathbb{K}}$ and its canonical dual $\{\tilde{g_k}\}_{k\in\mathbb{K}}$ are biorthonormal, i.e.,

$$\langle g_m, \tilde{g_k} \rangle = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}.$$

Conversely, if $\{g_k\}_{k\in\mathbb{K}}$ and $\{\tilde{g_k}\}_{k\in\mathbb{K}}$ are biorthonormal, then $\{g_k\}_{k\in\mathbb{K}}$ is exact.

Proof. $\{g_k\}_{k\in\mathbb{K}}$ is exact $\xRightarrow{\text{Thm. 1.35}}$ $\langle g_m, \tilde{g_m} \rangle = 1, \forall m \in \mathbb{K}$

$\xRightarrow{\text{Lem. 1.34}}$ $\langle g_m, \tilde{g_k} \rangle = 0, \forall k \neq m$

biorthonormality $\Rightarrow$ exact

$$\langle g_m, \tilde{g}_m \rangle = 1, \ \forall m \in K \overset{\text{Thm. 1.35}}{\Longrightarrow} \{g_k\}_{k \in K} \text{ is exact}$$

Thm. 1.37. If $\{g_k\}$ is an exact frame for $\mathcal{H}$ and $x = \sum_k c_k g_k$ with $x \in \mathcal{H}$, then the c_k are unique and given by

$$c_k = \langle x, \tilde{g}_k \rangle$$

where $\{\tilde{g}_k\}_{k \in K}$ is the canonical dual of $\{g_k\}_{k \in K}$.

Proof.

$$x = \sum_k \langle x, \tilde{g}_k \rangle g_k$$

$$x = \sum_k c_k g_k$$

$$\langle x, \tilde{g}_m \rangle = \left\langle \sum_k c_k g_k, \tilde{g}_m \right\rangle$$

$$= \sum_k c_k \underbrace{\langle g_k, \tilde{g}_m \rangle} = c_m$$

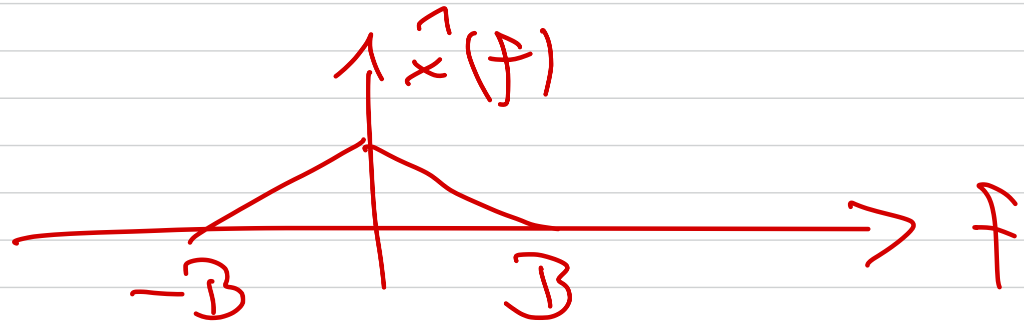
$$= \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases} .$$

1.4. Sampling Theorem

$$x(t) \circ\!\!-\!\!\bullet\ \hat{x}(f) = \int x(t)\, e^{-i2\pi f t}\, dt = \langle x(\cdot), e^{i2\pi f \cdot} \rangle$$

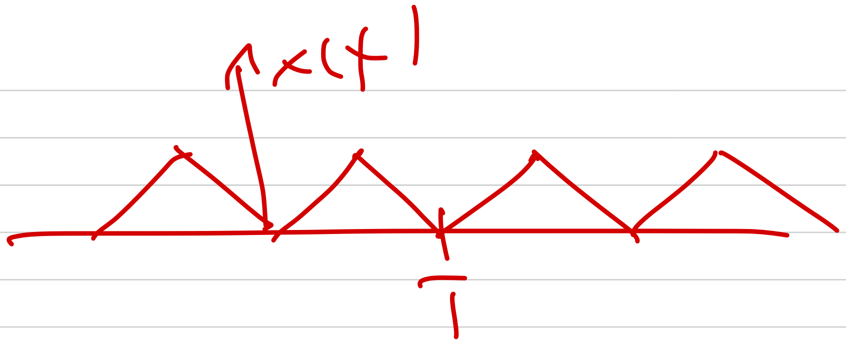
$$x(t) = \int \underbrace{\hat{x}(f)}_{\langle x(\cdot),\, e^{i2\pi f \cdot} \rangle} e^{i2\pi f t}\, df = \int \langle x(\cdot), e^{i2\pi f \cdot} \rangle\, e^{i2\pi f t}\, df$$

$x(t)$ is B-bandlimited if $\hat{x}(f) = 0$ for $|f| \geq B$



$$\underbrace{\sum_k x(t + kT)}_{y(t)} = \frac{1}{T} \sum_k \hat{x}\left(\frac{k}{T}\right) e^{i2\pi t \frac{k}{T}}$$

Poisson summation formula



$$y(t) = y(t+T) = \sum_k x(t+kT+T) = \sum_k x(t+(k+1)T)$$

$$k' = k+1$$

$$= \sum_{k'} x(t+k'T) = y(t)$$

$$c_\ell = \frac{1}{T}\int_0^T \sum_k x(t+kT)\, e^{-i2\pi\ell\frac{t}{T}}\,dt$$

$$= \frac{1}{T}\sum_k \int_0^T x(t+kT)\, e^{-i2\pi\ell\frac{t}{T}}\,dt =$$

$$t' = t+kT$$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t')\, e^{-i2\pi\ell\frac{t'-kT}{T}}\,dt'$$

$$= \frac{1}{T}\sum_k \int_{-kT}^{-kT+T} x(t)\, e^{-i2\pi\ell\frac{t}{T}}\,dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi \ell \frac{t}{T}}\, dt = \frac{1}{T} \hat{x}\left(\frac{\ell}{T}\right)$$

$$\boxed{\begin{aligned} \hat{x}_d(f) &= \sum_{k=-\infty}^{\infty} x(kT)\, e^{-i2\pi k f} \\ &\overset{P.S.F.}{=} \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{x}\left(\frac{f+k}{T}\right) \end{aligned}}$$

(DTFT)

$$\frac{f}{T} = B \Rightarrow f = BT$$

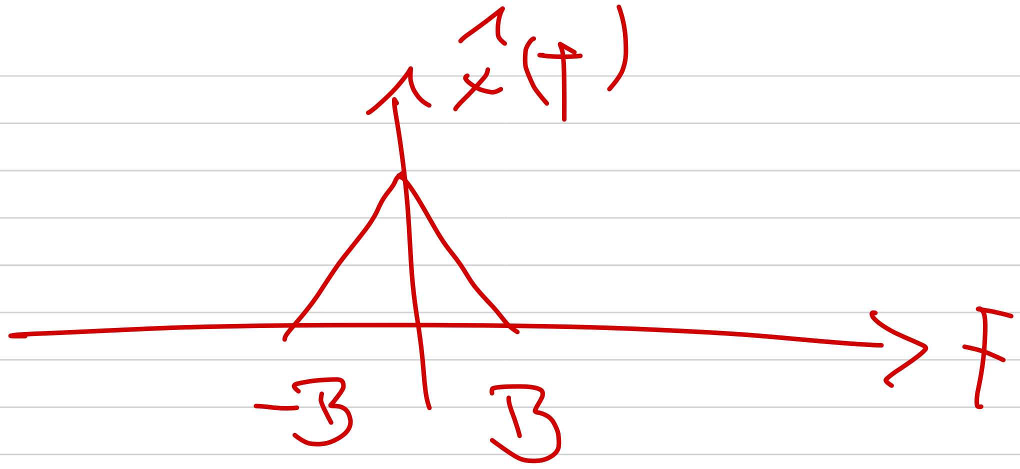
P.S.F. $\quad \sum_k x(k) = \sum_k \hat{x}(k)$

$$\uparrow$$

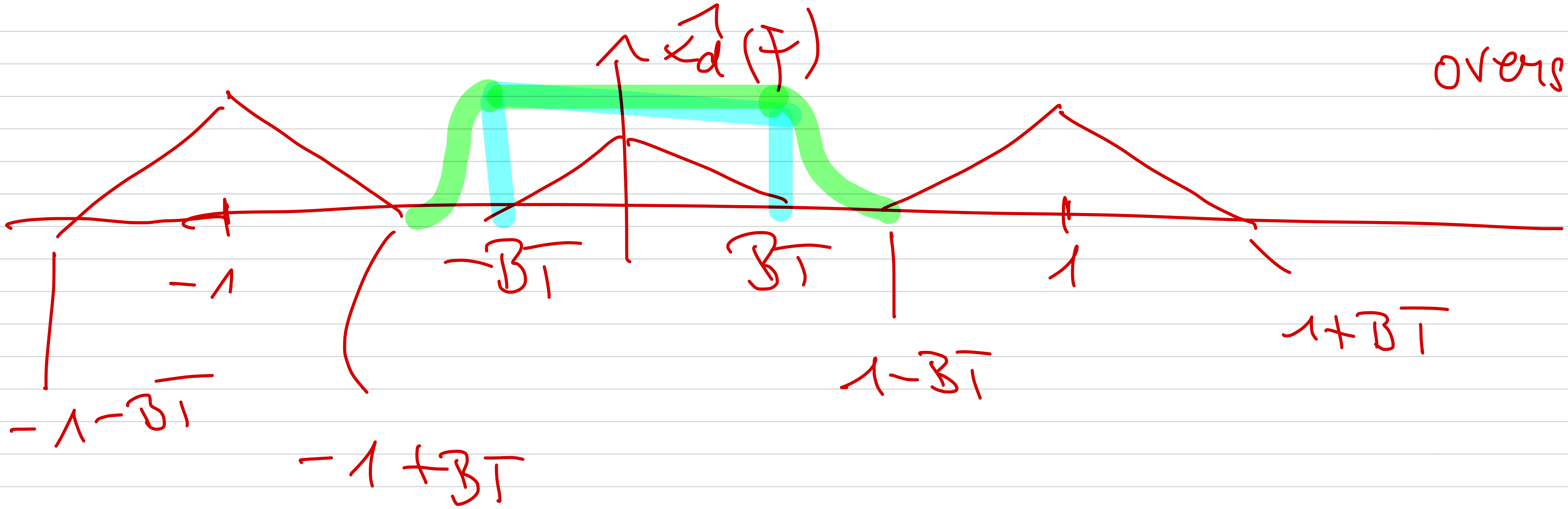
$$x(t \cdot T)\, e^{-i2\pi t f}$$

$k = -1$: $\quad \frac{f-1}{T} = -B$

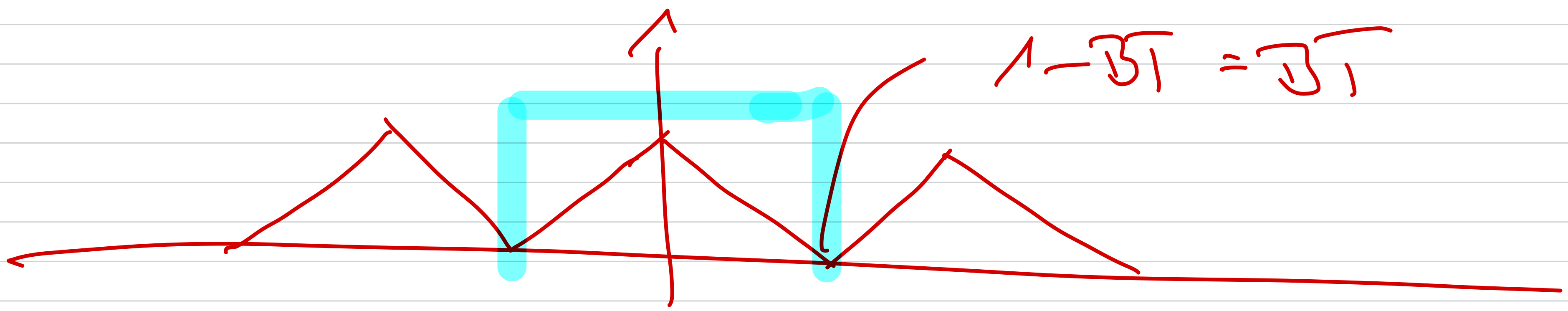
$$f = 1 \pm BT$$

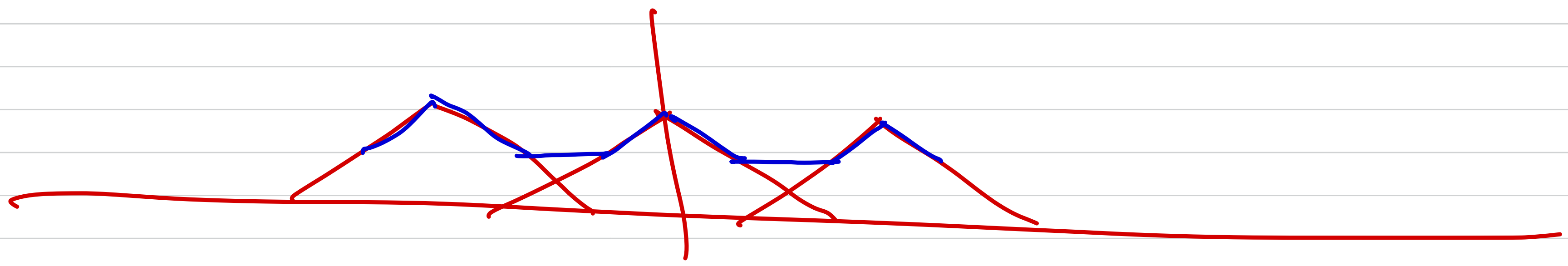


oversampled case



critical sampling





undersampling

$1 - BT < BT$

$1 - BT \geq BT$

$1 \geq 2BT$

$\frac{1}{T} \geq 2B$

$\uparrow$

$f_s = \frac{1}{T}$ ... sampling rate

$$\hat{x}_d(f)\,T\,\hat{h}_{LP}(f) = \hat{x}(f/T) \quad \Big| \quad f \to fT$$

$$\hat{x}(f) = \hat{x}_d(fT)\,T\,\hat{h}_{LP}(fT)$$

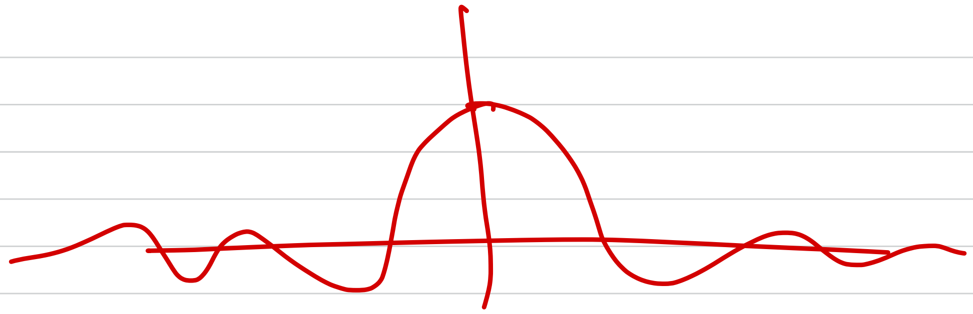
$$\hat{h}_{LP}(f) = \begin{cases} 1, & |f| \leq BT \\ 0, & \text{else} \end{cases}$$

$$\hat{x}(f) = \underbrace{T\,\hat{h}_{LP}(fT) \sum_{k} x(kT)\, e^{-i2\pi k f}}$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\, e^{i2\pi f t}\, df$$

$$= 2BT \sum_{k=-\infty}^{\infty} x(kT)\, \operatorname{sinc}(2B(t-kT))$$

$$\operatorname{sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}$$



Thm. 1.38. Sampling theorem. Let $x \in \mathcal{L}^2(B)$. Then, $x(t)$ is uniquely specified by its samples $x(kT)$, $k \in \mathbb{Z}$, if $\frac{1}{T} \geq 2B$. Specifically, we can reconstruct $x(t)$ according to

$$x(t) = 2BT \sum_{k} x(kT)\, \mathrm{sinc}(2B(t-kT)).$$

1.4.1. Sampling theorem as a frame expansion

$$g_k(t) = 2B\, \mathrm{sinc}(2B(t-kT))$$

$$x(kT) = \int_{-B}^{B} \hat{x}(f)\, e^{i2\pi kTf}\, df = \langle \hat{x}, \hat{g_k} \rangle = \langle x, g_k \rangle$$

$$\hat{g_k}(f) = \begin{cases} e^{-i2\pi kTf}, & |f| \leq B \\ 0, & \text{else} \end{cases}$$

$$x(t) = T \sum_{k} \langle x, g_k \rangle g_k(t)$$

with $g_k(t) = 2B \operatorname{sinc}(2B(t - kT))$

$$\|x\|^2 = \langle x, x \rangle = \langle T \sum_{k} \langle x, g_k \rangle g_k, x \rangle$$

$$= T \sum_{k} |\langle x, g_k \rangle|^2$$

$$\frac{1}{T} \|x\|^2 = \sum_{k} |\langle x, g_k \rangle|^2 = \langle Sx, x \rangle$$

$\langle Sx, x \rangle = \frac{1}{T} \|x\|^2 \Rightarrow$ tight with $A = \frac{1}{T}$

$$T : x \Rightarrow \{ \langle x, g_k \rangle \}_{k \in \mathbb{Z}}$$

$$T^* : \{c_k\}_{k \in \mathbb{Z}} \Rightarrow \sum_{k} c_k g_k$$

$$\tilde{g}_k = S^{-1} g_k = T g_k$$

$$S = \frac{1}{T} \mathbb{I}$$

$$\langle g_k, \tilde{g}_k \rangle = \langle g_k, T g_k \rangle = T \|g_k\|^2 = T \|\hat{g}_k\|^2 = 2BT$$

critical s.

$$f_s = \frac{1}{T} = 2B \Rightarrow 2BT = 1$$

crit. samp. $\langle g_k, \tilde{g}_k \rangle = 2BT = 1$, $\forall k \in \mathbb{Z} \Rightarrow$ exact

$$2BT = \frac{2B}{\frac{1}{T}} = \frac{2B}{f_s}$$

$$g_k'(t) = \sqrt{T}\, g_k$$

$$x(t) = T \sum_k \langle x, g_k \rangle g_k = \sum_k \langle x, g_k' \rangle g_k')$$

$$\Rightarrow A = 1$$

$$\|g_k'\|^2 = T\|g_k\|^2 = T\,\|\hat{g}_k\|^2 = 2BT \overset{\text{c.s.}}{=} 1$$

$\Rightarrow$ ONB in the case of c.s.

$$\langle g_k, g_k\rangle = 2BT = \frac{2B}{f_S} \overset{\text{u.s.}}{\neq} 1 \Rightarrow \{g_k\}_{k\in\mathbb{Z}} \text{ is inexact}$$

Chapter 2. Uncertainty relations and sparse signal recovery

frequency extent

$$\sigma_t \; \sigma_f \geq \text{const.}$$

time-duration

$$x(at) \circ\!\!-\!\!\bullet \frac{1}{|a|}\hat{x}(f/a)$$

$$x(t) = \int \hat{x}(f)\, e^{i2\pi ft}\, df$$

$$\hat{x}(f) = \int x(t)\, e^{-i2\pi ft}\, dt$$

$$x(t) = \int \langle x(\cdot), e^{2\pi i f\cdot}\rangle e^{i2\pi ft}\, df$$

$$x(t) = \int x(t')\,\delta(t-t')\,dt'$$

$$= \langle x(\cdot), \delta(t-\cdot)\rangle$$

$$x(t) =$$

$$x = \sum_i \langle x, e_i \rangle e_i$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \ldots$$

$$x = \sum_i \langle x, f_i \rangle f_i$$

$$F = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & \text{—} & 1 \\ 1 & \omega & \omega^2 & \ldots \\ \vdots & \omega^2 & \ldots & \\ 1 & ? & & \end{bmatrix}$$

$$\omega = e^{-i2\pi/m}$$

$$f_1 \quad f_2 \quad \ldots$$

$$F F^H = F^H F = I_m$$

Notation. U -- unitary

$$P_{\mathcal{A}}(U) = U D_{\mathcal{A}} U^H \qquad \mathcal{A} = \{1, 3, 7, \ldots\}$$

$$D_{\mathcal{A}} = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}, \quad D_{\mathcal{A}} x = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ \vdots \\ x_7 \end{pmatrix}$$

$$P_{\mathcal{A}}(U) = \sum_{i \in \mathcal{A}} u_i u_i^H, \quad U = F \text{ DFT matrix}$$

$$W^{U, \mathcal{A}} = R(P_{\mathcal{A}}(U))$$

$$x_{\mathcal{A}} = D_{\mathcal{A}} x$$

$$|||A|||_2 = \max_{x : \|x\|_2 = 1} \|Ax\|_2 \quad \text{op. 2-norm}$$

$$(Ax)^H Ax = x^H A^H A x$$

$$\|A\|_2 = \sqrt{\operatorname{tr}(A A^H)} = \sqrt{\sum_{i,j} |A_{ij}|^2}$$

$$\underset{\substack{\uparrow \\ \text{e.g.f}}}{\Delta_{P,Q}(U)} = |||D_P \, P_Q(U)|||_2$$

Lemma 2.20.

$$\Delta_{P,Q}(U) = \max_{x \in W^{U,Q} \setminus \{0\}} \frac{\|xP\|_2}{\|x\|_2}$$

$AA^H = I$

$$D_P = I D_P I^H, \quad P_P(A) = A D_P A^H$$

$BB^H = I$

$$P_Q(U) = U D_Q U^H, \quad P_Q(B) = B D_Q B^H$$

$$|||P_P(A) P_Q(B)|||_2 = |||A D_P A^H B D_Q B^H|||_2$$

$$= |||A^H A D_P A^H B D_Q B^H|||_2$$

Ax

$x : \|x\|_2 = 1$

A

$$= |||D_P \underbrace{A^H B}_{U} D_Q \underbrace{B^H}_{U^H}|||_2$$

$$= |||D_P U D_Q U^H|||_2 = |||D_P P_Q(U)|||_2$$

An uncertainty relation is an inequality of the form

$$\Delta_{P,Q}(U) \le c < 1.$$

In principle the norm $|||D_P P_Q(U)|||_2$ is difficult to quantify

later: In the case $U = F$, we can get an explicit expression for $|||D_P P_Q(U)|||_2$, but this is an absolute exception

$$\frac{\|D_P P_Q(U)\|_2}{\sqrt{\mathrm{rank}(D_P P_Q(U))}} \le \Delta_{P,Q}(U) \le \|D_P P_Q(U)\|_2 = \sqrt{\mathrm{tr}(D_P \underbrace{P_Q(U) P_Q(U)}_{P_Q(U)} D_P)} = \sqrt{\mathrm{tr}(D_P P_Q(U))}$$

$$\mathrm{rank}(D_P P_Q(U)) = \mathrm{rank}(\underline{D_P}\, \underline{U D_Q U^H}) \le \min(|P|, |Q|)$$

$$\sqrt{\frac{\mathrm{tr}(D_P P_Q(U))}{\min(|P|,|Q|)}} \le \Delta_{P,Q}(U) \le \sqrt{\mathrm{tr}(D_P P_Q(U))}$$

$F \in \mathbb{C}^{n \times n}$

$$[F]_{k,l} = F_{k,l} = \frac{1}{\sqrt{n}} e^{i \frac{k l 2\pi}{n}}$$

particularize to $U = F$, and get

$$\sqrt{\mathrm{tr}(D_P P_Q(F))} = \sqrt{\mathrm{tr}(D_P F D_Q F^H)} = \sqrt{\sum_{i \in P} \sum_{j \in Q} |F_{i,j}|^2} = \sqrt{\frac{|P||Q|}{n}}$$

$$\sqrt{\frac{\max(|P|,|Q|)}{m}} \le \Delta_{P,Q}(F) \le \sqrt{\frac{|P||Q|}{m}}$$

Ex. 1. $P = \{1\}, \quad Q = \{1, \ldots, m\}$

$$1 \le \Delta_{P,Q}(F) \le 1 \Rightarrow \Delta_{P,Q}(F) = 1$$

Ex. 2. $P = \left\{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \right\}, \quad n \text{ div. } m$

$Q = \{ \ell+1, \ldots, \ell+n \}$

$\ell \in \{1, \ldots, m\}$

$$\sqrt{\frac{n}{m}} \le \Delta_{P,Q}(F) \le \sqrt{\frac{n^2}{m}} = \frac{n}{\sqrt{m}}$$

$$\sqrt{\frac{n}{m}} \le \Delta_{P,Q}(F) \le \frac{n}{\sqrt{m}} = \sqrt{n}\,\underbrace{\sqrt{\frac{n}{m}}}_{\text{const.}}$$

Lemma 2.1. Let n divide m and consider

$$P = \left\{ \frac{m}{n}, \frac{2m}{n}, \ldots, \frac{(n-1)m}{n}, m \right\}$$

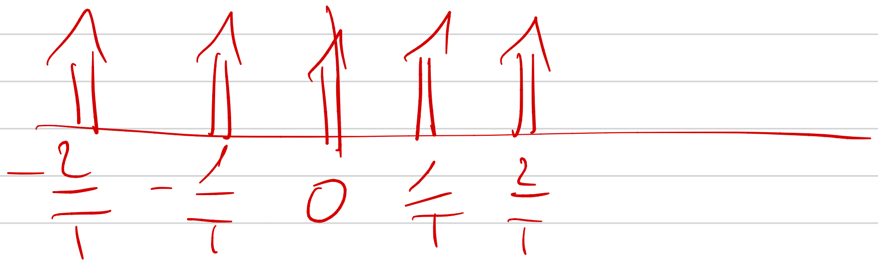
$$Q = \{ l+1, \ldots, l+n \}, \quad Q \text{ interpreted circularly}$$

Then, $\Delta_{P,Q}(F) = \sqrt{n/m}$.

Proof uses a discrete version of the following:

proof is highly specific to the Fourier transform

$$\sum_{l} \delta(t - lT) \circ\!\!-\!\!\bullet \frac{1}{T} \sum_{l} \delta\left(f - \frac{l}{T}\right)$$



Large-sieve:

$$\Delta_{P,Q}(F) \le \sqrt{2}\sqrt{n/m}$$

$$\left(\sqrt{\frac{n}{m}} \le \Delta_{P,Q}(F) \le \sqrt{n}\sqrt{\frac{n}{m}} \right)$$

2.2.2. Coherence-based uncertainty relations

Def. 2.3. For $A=(a_1,\dots,a_n)\in\mathbb{C}^{m\times n}$ with columns $\|\cdot\|_2$-normalized to 1, the coherence of A is defined as

$$\mu(A) := \max_{i\neq j} |a_i^H a_j| .$$

Lemma 2.4. Let $U\in\mathbb{C}^{m\times m}$ be unitary and $P,Q\subseteq\{1,\dots,m\}$. Then,

$$\Delta_{P,Q}(U) \le \sqrt{|P||Q|}\,\mu([I\ U]).$$

Proof.

$$\Delta_{P,Q}^2(U) \le \mathrm{tr}(D_P\, U D_Q U^H)$$

$$\left(\mathrm{tr}(D_P\, \underbrace{P_Q(U) P_Q(U)}_{P_Q(U)} D_P)\right)$$

$$= \sum_{q\in P}\sum_{l\in Q} |U_{q,l}|^2$$

$$\le |P|\,|Q| \max_{q,l} |U_{q,l}|^2$$

$$= |P||Q|\,\mu^2\left([I\ U]\right). \quad \square$$

2.2.3. Concentration inequalities

$$(s :=)\, p = \overline{F} q$$

p ← freq.-domain signal; q ← time-domain signal; $\overline{F}$ ↑ DFT

$$p = \overline{F} q$$

$$s := \overline{I} p = \overline{F} q \qquad s := Ap = Bq$$

$$\overline{I} \Rightarrow A \qquad Ap = Bq \,|\, A^H.$$

$$\overline{F} \Rightarrow B \qquad p = \underbrace{A^H B}_{U} q = Uq$$

$$(s :=)\ \boxed{p = Uq}\ , \ U \text{ unitary}$$

Def. 2.5. Let $P \subseteq \{1, \dots, m\}$ and $\varepsilon_P \in [0, 1]$. The vector $x \in \mathbb{C}^m$ is said to be ε_P-concentrated if $\|x - x_P\|_2 \leq \varepsilon_P \|x\|_2$.

$$\frac{\|x - x_P\|_2}{\|x\|_2} \le \varepsilon_P$$

$$\varepsilon_P = 0 \implies \frac{\|x - x_P\|_2}{\|x\|_2} = 0 \implies x = x_P \implies x \text{ is strictly supported on } P$$

Lemma 2.6. Let $U \in \mathbb{C}^{m\times m}$ be unitary and $P, Q \subseteq \{1, \dots, m\}$. Suppose that there exists a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $p = Uq$. Then,

$$\Delta_{P,Q}(U) \ge [1 - \varepsilon_P - \varepsilon_Q]_+ .$$

Proof. We have

$$p - P_Q(U)_p + P_Q(U)_p - P_Q(U)p_P$$

$$\|p - P_Q(U)p_P\|_2 \le \|p - P_Q(U)p\|_2 + \|P_Q(U)p_P - P_Q(U)p\|_2$$

$$\le \|p - P_Q(U)p\|_2 + \|\|P_Q(U)\|\|_2 \, \|p_P - p\|_2$$

$$\le \|p - P_Q(U)p\|_2 + 1 \cdot \|p_P - p\|_2$$

$$= \|p - U D_Q \overbrace{U^H p}^{q}\|_2 + \|p_P - p\|_2$$

$$(q = U^H p)$$

$$= \|q - q_Q\|_2 + \|p - p_P\|_2$$

$$\overset{\text{Def. 2.5.}}{\leq} \varepsilon_Q \|q\|_2 + \varepsilon_P \|p\|_2 \quad (= U^H p)$$

$$= (\varepsilon_P + \varepsilon_Q) \|p\|_2 .$$

$$P_Q(U) p_P = p - p + P_Q(U) p$$

$$\| P_Q(U) p_P \|_2 \overset{\text{rev. } \Delta\text{-inequ.}}{\geq} [\|p\|_2 - \underbrace{\| p - P_Q(U) p_P \|_2}_{\leq (\varepsilon_P + \varepsilon_Q)\|p\|_2}]_+$$

$$\geq \|p\|_2 [1 - (\varepsilon_P + \varepsilon_Q)]_+ \quad \Big| \; \frac{1}{\|p\|_2}$$

$$\| P_Q(U) D_P \frac{p}{\|p\|_2} \|_2 \geq [1 - \varepsilon_P - \varepsilon_Q]_+ .$$

$$\Delta_{P,Q}(U) \geq [1 - \varepsilon_P - \varepsilon_Q]_+ . \qquad \square$$

Corollary 2.7. Let $A, B \in \mathbb{C}^{m \times m}$ and $P, Q \subseteq \{1, \ldots, m\}$. Suppose that there exist a nonzero ε_P-concentrated $p \in \mathbb{C}^m$ and a nonzero ε_Q-concentrated $q \in \mathbb{C}^m$ s.t. $Ap = Bq$. Then,

$$|P||Q| \geq \frac{[1-\varepsilon_P-\varepsilon_Q]_+^2}{\mu^2([A\ B])}.$$

Proof. Let $U = A^H B$. Then,

$$[1-\varepsilon_P-\varepsilon_Q]_+ \overset{L2.6.}{\leq} \Delta_{P,Q}(U) \overset{L2.4.}{\leq} \sqrt{|P||Q|}\ \mu([I\ U])$$

$$\mu([I\ U]) = \mu([I\ A^H B]) = \mu([A\ B]).\ \square$$

1.st special case: $\varepsilon_P = \varepsilon_Q = 0 \Rightarrow$ Elad–Bruckstein result, 2003

Corollary 2.8. Let $A, B \in \mathbb{C}^{m\times m}$ be unitary. If $Ap = Bq$ for nonzero $p, q \in \mathbb{C}^m$, then

$$\|p\|_0 \|q\|_0 \geq \frac{1}{\mu^2([A\ B])}.$$

2.nd special case: $\overset{A=I,}{U = F} \Rightarrow \mu([I\ F]) = \frac{1}{\sqrt{m}}$

$$\|p\|_0 \|q\|_0 \geq m. \quad \text{Donoho–Stark, 1989}$$

$$p = Uq = Fq$$

2.2.4. Noisy recovery in $(\mathbb{C}^m, \|\cdot\|_2)$

$p, n \in \mathbb{C}^m$

$P \subseteq \{1, \dots, m\}$

$P^c = \{1, \dots, m\} \setminus P$

observe

$$y = p_{P^c} + n$$

the samples indexed by P are lost in the observation and noise gets added

Q: Can we recover p from y?

In general, no!

However, if p satisfies certain structural properties and $|P|$ is not too large, then this is, indeed, possible.

sparsity

Lemma 2.9. Let $U \in \mathbb{C}^{m\times m}$ be unitary, $Q \subseteq \{1,\dots,m\}$, $p \in \mathcal{W}^{U,Q}$, and consider

(discard the elements of p supported on P)

$$y = P_{P^c}\,p + n, \quad \text{add noise}$$

where $n \in \mathbb{C}^m$ and $P^c = \{1,\dots,m\}\setminus P$ with $P \subseteq \{1,\dots,m\}$. If

$$\left(\Delta_{P,Q}(U) = |||D_P P_Q(U)|||_2\right) \quad \Delta_{P,Q}(U) < 1, \quad \text{recovery condition}$$

then there exists a matrix $L \in \mathbb{C}^{m\times m}$ s.t.

all the noise we see

$$\|Ly - p\|_2 \leq C\,\|n_{P^c}\|_2$$

with $C = \dfrac{1}{(1-\Delta_{P,Q}(U))}$.

In particular,

$$|P|\,|Q| < \frac{1}{\mu^2(U)}$$

is sufficient for $\Delta_{P,Q}(U) < 1$.

follows by application of L.2.4.

Proof. For $\Delta_{P,Q}(U) < 1$, it follows that $(I - D_P P_Q(U))$ is inv. with

$$|||(I - D_P P_Q(U))^{-1}|||_2 \leq \frac{1}{1-|||D_P P_Q(U)|||_2} = \frac{1}{1-\Delta_{P,Q}(U)}$$

(Neumann series expansion)

$$L = (I - D_p P_{\Omega}(u))^{-1} D_{pc}$$

$$L\, p_{pc} = (I - D_p P_{\Omega}(u))^{-1} \underbrace{D_{pc}\, p}_{p_{pc}}$$

$p \in W^{u,0}$

$$= (I - D_p P_{\Omega}(u))^{-1} (I - D_p) p$$

$$= (I - D_p P_{\Omega}(u))^{-1} (I - D_p P_{\Omega}(u)) p$$

$$= p$$

$$\| Ly - p \|_2 = \| L(p_{pc} + n) - p \|_2$$

$$= \| Ln \|_2$$

$$\leq \| (I - D_p P_{\Omega}(u))^{-1} \|_2 \, \| n_{pc} \|_2$$

$$\leq \frac{1}{1 - \Delta_{p,\Omega}(u)} \| n_{pc} \|_2 \, . \quad \square$$

Note: In the noise-free case, $\Delta_{P,Q}(U) < 1 \Rightarrow$ perfect recovery.

$P = \{\frac{m}{n}, \frac{2m}{n}, \dots, \frac{(n-1)m}{n}, m\}$ ← indicates what we throw away

$Q = \{l+1, \dots, l+n\}$ ← carries p

$U = F$

p is n-sparse in $U = F$

we are missing n entries in the noisy observation y

$\Delta_{P,Q}(F) = \sqrt{n/m}$ (L.7.1)

Stable recovery of p is possible for $n \le m/2$

$m \ge 2n$

$\Delta_{P,Q}(F) \le \frac{n}{\sqrt{m}} < 1$

$n < \sqrt{m}$

$m > n^2 \Rightarrow$ square-root bottleneck

Compressive sensing

$m \propto s^2$ $\sqrt{\cdot}$ bottleneck

$s < \frac{1}{2}\left(1 + \frac{1}{\mu}\right)$, (P0), (P1)

μ --- coherence

$\mu = \frac{1}{\sqrt{m}}$, $D = [I \ \ F]$

$s < \frac{1}{2}\left(1 + \sqrt{m}\right)$

$s \sim \sqrt{m}$

$\boxed{m \propto s^2}$, $n = 1024$

$s = 30$

$m \propto 900$ measurements

Theorem 3.8. Let $D \in \mathbb{C}^{m \times n}$ be a dictionary with coherence $\mu(D)$

Then, we have

$$\mu(D) \geq \sqrt{\frac{n-m}{m(n-1)}}$$

where $m \leq n$.

Proof. Set $G = D^H D \in \mathbb{C}^{n \times n}$. Then, G has the following properties

1. G has ones along the main diagonal

$$\left(D = [d_1 \dots d_n], \quad G = D^H D = \begin{bmatrix} d_1^H \\ d_2^H \\ \vdots \end{bmatrix} [d_1 \; d_2 \dots] = \begin{bmatrix} \|d_1\|^2 & \langle d_1, d_2\rangle^* & \dots \\ \langle d_1, d_2\rangle & \|d_2\|^2 & \\ \vdots & & \end{bmatrix} \right)$$

2. G is positive semi-definite with rank (at most) m.

$$x^H D^H D x = \|Dx\|^2 \geq 0$$

$\lambda = (\lambda_1, \dots, \lambda_m)^T$ is the vector of nonzero eigenvalues of G.

$$\operatorname{tr}(G) = \sum_{i=1}^{m} \lambda_i = \|\lambda\|_1 = n$$

$$\|G\|_2^2 = \mathrm{tr}(G^H G) = \sum_{i=1}^{m} \lambda_i^2 = \|\lambda\|_2^2$$

Jensen's inequality

$$\overset{(Ex)^2}{f(Ex)} \le E(\overset{x^2}{f(x)})$$

$$\left(\frac{1}{m}\sum_{i=1}^{m} \lambda_i\right)^2 \le \frac{1}{m}\sum_{i=1}^{m} \lambda_i^2$$

$$\left(\frac{1}{m}\|\lambda\|_1\right)^2 \le \frac{1}{m}\|\lambda\|_2^2$$

$$\|\lambda\|_1^2 \le m\,\|\lambda\|_2^2 \quad \|G\|_2^2$$

$$\|G\|_2^2 \ge \frac{1}{m}\|\lambda\|_1^2 = \frac{n^2}{m}$$

$$\|G\|_2^2 = n + \sum_{i=1}^{n}\sum_{j\neq i} |\langle d_i, d_j\rangle|^2 \ge \frac{n^2}{m}$$

$$\mu(D)^2 \ge \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i} |\langle d_i, d_j\rangle|^2 \leftarrow \text{def. of coherence}$$

$$\geq \frac{1}{n(n-1)}\left(\frac{n^2}{m} - n\right)$$

$$= \frac{n-m}{m(n-1)}$$

$$\Rightarrow \quad \mu(D) \geq \sqrt{\frac{n-m}{m(n-1)}} \quad . \quad \square$$

Interpretation:

$$\mu(D) \geq \sqrt{\frac{n-m}{m(n-1)}}, \qquad n \gg m$$

$$\uparrow$$

$$\sim \sqrt{\frac{n}{n \cdot m}} = \frac{1}{\sqrt{m}}$$

$$\mu(D) \geq \frac{1}{\sqrt{m}}$$ Welch-bound inequality

$$S < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right) \Rightarrow \sqrt{\cdot}$$ – bottleneck

Q: Can we improve upon the scaling behavior $m \gtrsim s^2$?

Chapter 4

Finite Rate of Innovation

(recall: s-sparse signal x_1

$y_1 = D x_1$

s-sparse signal x_2

$y_2 = D x_2$ $\Bigg\}$ $x_1 \neq x_2$

$$0 \neq y_1 - y_2 = D(\underbrace{x_1 - x_2}_{(2s)\text{-sparse}})$$

$2s$ is the minimum no. of measurements needed to be able, in principle, to recover an s-sparse signal

(P0) recovery from $2s$ measurements, but need to verify that spark$(D) = 2s+1$

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right) \Rightarrow \sqrt{\cdot} \text{ - bottleneck}$$

(P1) alg. feasible, convex-opt. based

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(D)}\right) \Rightarrow \sqrt{\cdot} \text{ - bottleneck}$$

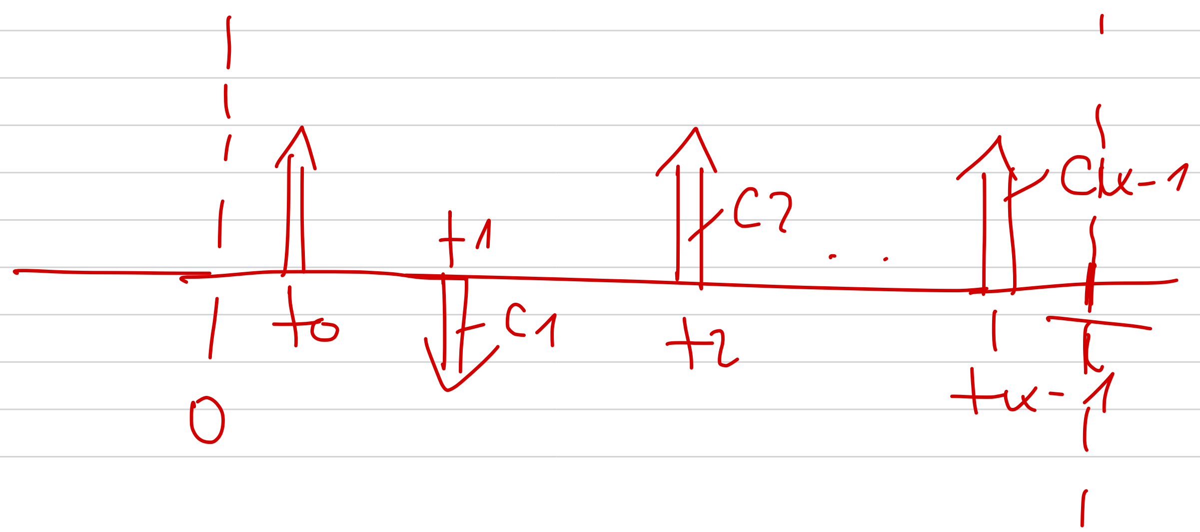
$$x(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k), \quad 0 \le t_k \le \tau$$

$\delta(t)$ ... Dirac delta distribution

K unknowns $\{t_k\}$

K coefficients $\{c_k\}$

recover from measurements of $x(t)$



We will derive an algorithm which recovers $\{t_k\}$ & $\{c_k\}$ from $2K+1$ measurements.

periodically extend $x(t)$ to a τ- periodic signal and compute its Fourier series coefficients, which amounts to taking measurements in the frequency domain or in the DFT basis.

$$d_n = \frac{1}{\tau} \int_0^{\tau} \sum_{k=0}^{K-1} c_k \, \delta(t-t_k) \, e^{-i2\pi \frac{n}{\tau} t} \, dt \qquad \left(d_n = \frac{1}{\tau} \int_0^{\tau} x(t) \, e^{-i2\pi \frac{n}{\tau} t} dt \right)$$

$$= \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{\int_0^{\tau} e^{-i2\pi \frac{n}{\tau} t} \, \delta(t-t_k) \, dt}_{\text{sifting property}}$$

sifting property $\left(\int f(t) \, \delta(t-t_0) \, dt = f(t_0) \right)$

$$= \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \, e^{-i2\pi \frac{n}{\tau} t_k}$$

$$x(t) = \sum_{n\in\mathbb{Z}} d_n \, e^{i2\pi \frac{n}{\tau} t_k} = \sum_{n\in\mathbb{Z}} \left(\frac{1}{\tau} \sum_{k=0}^{K-1} c_k \, e^{-i2\pi \frac{n}{\tau} t_k} \right) e^{i2\pi n \frac{t}{\tau}}$$

The recovery algorithm (recover $\{t_k, c_k\}_{k=0}^{K-1}$ from d_n)

"annihilating filter method"

$\Rightarrow A(z) = \sum_{m=0}^{K} a_m z^{-m} \quad \longleftarrow$ z transform (Laurent series)

Filter: $x_m \;—\; \boxed{a_m} \;—\; y_m = \sum_{l=0}^{K} a_l \, x_{m-l}$

convolution operator with kernel $\{a_m\}_{m=0}^{K}$ $\in \mathbb{C}$

Property: $a_1 * a_2 \;\overset{z}{\circ\!\!-\!\!\!-\!\!\bullet}\; A_1(z) A_2(z)$

$$x_m \;—\; \underbrace{\boxed{a_2} \;—\; \boxed{a_1}}_{a_1 * a_2} \;—\; y_m$$

$$A(z) = \sum_{m=0}^{K-1} a_m z^{-m} = \prod_{k=0}^{K-1} \underbrace{\left(1 - e^{-i2\pi\frac{t_k}{\tau}} z^{-1}\right)}$$

$$\bullet\!\!-\!\!\circ \quad z$$

$$\delta[n] - e^{-i2\pi\frac{t_k}{\tau}} \delta[n-1]$$

$$\left(\overset{\circ\!-\!\bullet\; z}{1} - e^{-i2\pi\frac{t_k}{\tau}} \overset{\circ\!-\!\bullet}{z^{-1}} \right)$$

$$e^{-i2\pi\frac{t_k}{\tau}n} * \left(\delta[n] - e^{-i2\pi\frac{t_k}{\tau}} \delta[n-1] \right)$$

$$= e^{-i2\pi\frac{t_k}{\tau}n} - e^{-i2\pi\frac{t_k}{\tau}} e^{-i2\pi\frac{t_k}{\tau}(n-1)}$$

$$= e^{-i2\pi\frac{t_k}{\tau}n} - \underbrace{e^{-i2\pi\frac{t_k}{\tau}} e^{i2\pi\frac{t_k}{\tau}}}_{1} e^{-i2\pi\frac{t_k}{\tau}n} = \underline{\underline{0}}$$

$$d_n = \frac{1}{L} \sum_{k=0}^{K-1} c_k \, e^{-i 2\pi \frac{n}{L} t_k}$$

$$d_n * a_n = \left(\frac{1}{L} \sum_{k=0}^{K-1} c_k \, e^{-i 2\pi \frac{n}{L} t_k} \right) * \underbrace{\left(\delta[n] - e^{-i 2\pi \frac{t_0}{L}} \delta[n-1] \right) * \left(\delta[n] - e^{-i 2\pi \frac{t_1}{L}} \delta[n-1] \right) * \cdots * \left(\delta[n] - e^{-i 2\pi \frac{t_{K-1}}{L}} \delta[n-1] \right)}$$

$$= \frac{1}{L} c_0 \, e^{-i 2\pi \frac{n}{L} t_0} * \; - \; - \; -$$

$$+ \frac{1}{L} c_1 \, e^{-i 2\pi \frac{n}{L} t_1} * \; - \; -$$

$$\vdots$$

$$= \underline{\underline{0}}$$

$$\underline{d_n * a_n = 0}$$

$$\left(A(z) = \sum_{n=0}^{K} a_n z^{-n} = \prod_{k=0}^{K-1} \left(1 - e^{-i 2\pi \frac{t_k}{\tau}} z^{-1}\right) \right.$$

$t_k \in [0, \tau]$

$$\left. \text{zeros of } A(z) \text{ are given by } e^{-i 2\pi \frac{t_k}{\tau}} \Rightarrow t_k = -\frac{\tau}{2\pi} \arg(\text{zeros}) \right)$$

$$d_n * a_n = \sum_{l=0}^{K} d_l \, a_{n-l} = 0 \quad , \forall n \in \mathbb{Z}$$

rewrite $d_n * a_n = 0$ in matrix-vector form

$K+1$ equations for

$$\begin{array}{l} n=0 \rightarrow \\ n=1 \rightarrow \\ \\ n=K \rightarrow \end{array} \underbrace{\begin{bmatrix} & & \vdots & \\ d_0 & d_{-1} & \cdots & d_{-K} \\ d_1 & d_0 & \cdots & d_{-K+1} \\ & & \vdots & \\ d_K & d_{K-1} & \cdots & d_0 \end{bmatrix}}_{:= S} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix} = 0 \;, \quad S a = 0$$

$S \in \mathbb{C}^{(K+1)\times(K+1)}$, $a \in \mathbb{C}^{K+1}$

$k+1$

unknowns

S is a $(k+1)\times(k+1)$ matrix, it is built from $\{d_{-k},\dots,d_0,\dots,d_k\}$,

that is we need $2k+1$

Fourier series coefficients

no. of "measurements" $= 2k+1$

SVD of S, choose for a, the a as a singular vector corr. to

sing. value 0

assume for now uniqueness, then $A(z) = \sum_{m=0}^{k} a_m z^{-m}$

$$= \prod_{k=0}^{k-1} (1 - \alpha_k z^{-1})$$

ESPRIT

$$\alpha_k = e^{-i 2\pi \frac{t_k}{\tau}}$$

$$\left(y = D \begin{pmatrix} 0 \\ \vdots \\ x \\ 0 \\ \vdots \\ x \\ \vdots \end{pmatrix} = \bar{D}\,\bar{x} \quad \Rightarrow \text{solve for } \bar{x} \right)$$

4.1.2. Finding the c_ℓ

$$d_n = \frac{1}{L} \sum_{\ell=0}^{k-1} c_\ell\, u_\ell^{\,n}, \qquad u_\ell = e^{-i k_m \frac{t_\ell}{L}}$$

$$\frac{1}{L} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & & u_{k-1} \\ u_0^2 & u_1^2 & & \\ \vdots & \vdots & & \\ u_0^{k-1} & u_1^{k-1} & \cdots & u_{k-1}^{k-1} \end{bmatrix}}_{k \times k} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix}}_{\mathbb{C}^k} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \end{bmatrix}$$

full rank ? $\Rightarrow$ uniqueness

yes, by t_ℓ being pairwise distinct solve for c

4.1.3. Uniqueness in $d_n * a_n = 0$

$$d_n = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{e^{-i2\pi \frac{n}{\tau} k}}_{u_k^{\,n}}$$

$$\begin{bmatrix} d_0 & d_{-1} & \cdots & d_{-K} \\ d_1 & d_0 & \cdots & d_{-K+1} \\ & \vdots & & \\ d_K & d_{K-1} & \cdots & d_0 \end{bmatrix} = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \begin{bmatrix} 1 & u_k^{-1} & \cdots & u_k^{-K} \\ u_k & 1 & & \\ \vdots & & & \\ u_k^{K} & & \cdots & 1 \end{bmatrix}$$

S ... $(K+1)\times(K+1)$

$$= \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \begin{bmatrix} 1 \\ u_k \\ \vdots \\ u_k^{K} \end{bmatrix} \begin{bmatrix} 1 & u_k^{-1} & \cdots & u_k^{-K} \end{bmatrix}$$

$(k+1) \times (k+1)$ matrix
of rank 1

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots \\ a_0 & a_1 & \cdots \\ \vdots & \vdots & \\ a_0^k & a_1^k & \cdots \end{bmatrix}}_{\text{lin. ind. follows by Vandermonde property}}$$

$\operatorname{rank}(S) = k$

$(k+1) \times (k+1)$ matrix

$Sa = 0 \Rightarrow a$ must be unique. $\square$

measurement matrix

$y = Dx$

signal to be recovered

Fourier series coefficients

$$c_k = \frac{1}{\tau}\int_0^{\tau} x(t)\, e^{-i2\pi k \frac{t}{\tau}}\, dt$$

D

continuum

$2K+1$

$x(t)$ $\quad t \in [0, \tau]$

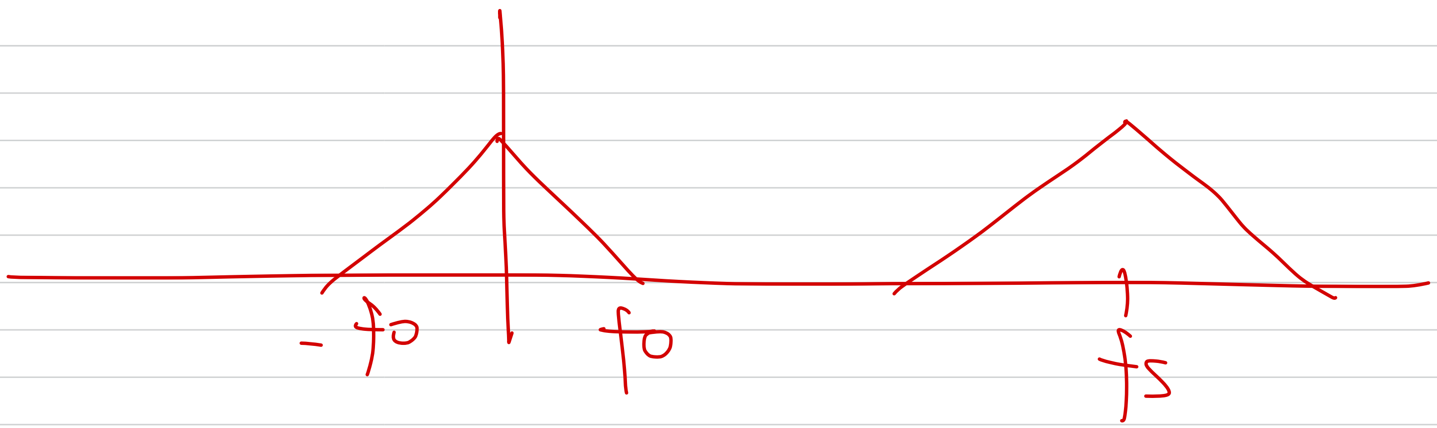
$$x(t) = \sum_{k=0}^{K} c_k\, \delta(t - t_k)$$

Chapter 5

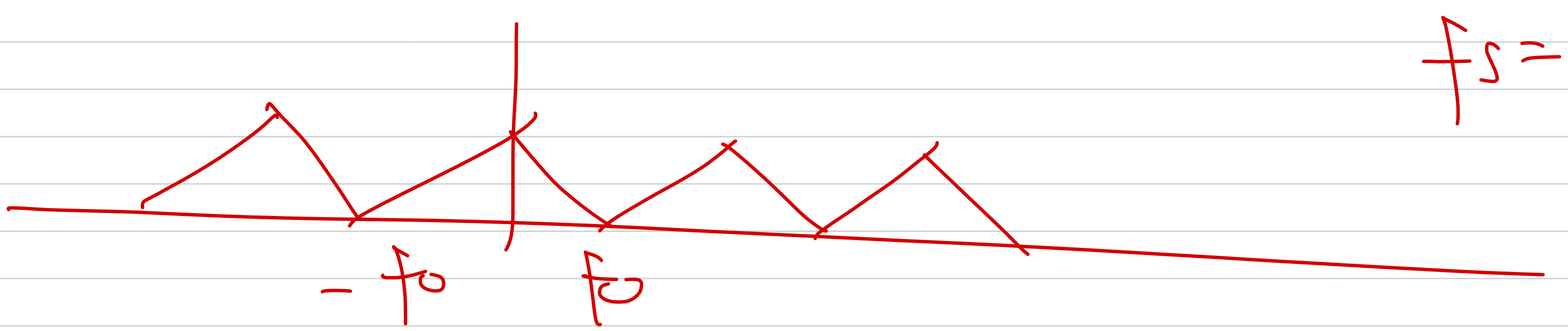
Sampling of multi-band signals

sparsity in the frequency-domain

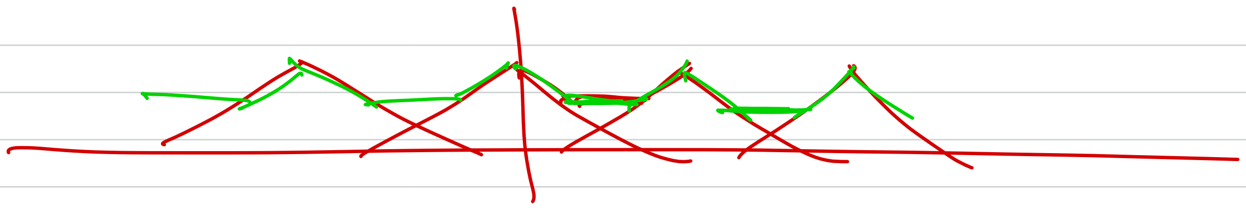
start with sampling theorem



$f_s > 2f_0$: oversampling



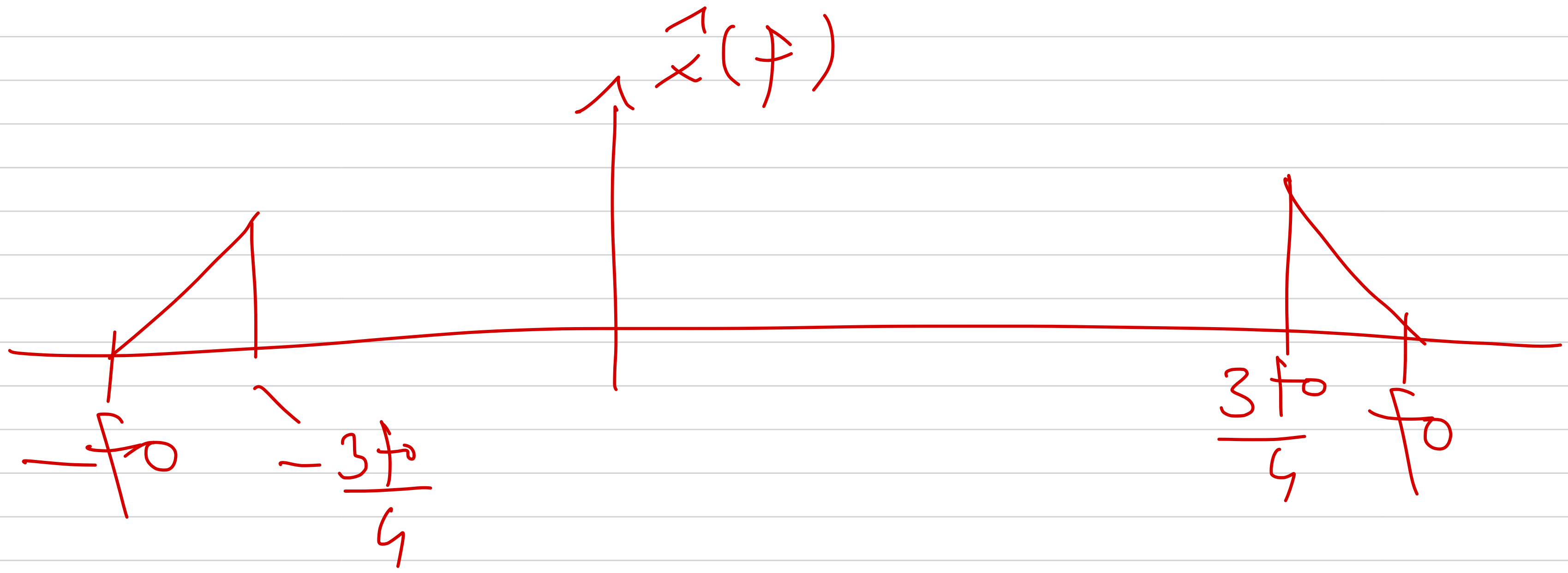
$f_s = 2f_0$: critical sampling



$f_s < 2f_0$ undersampling

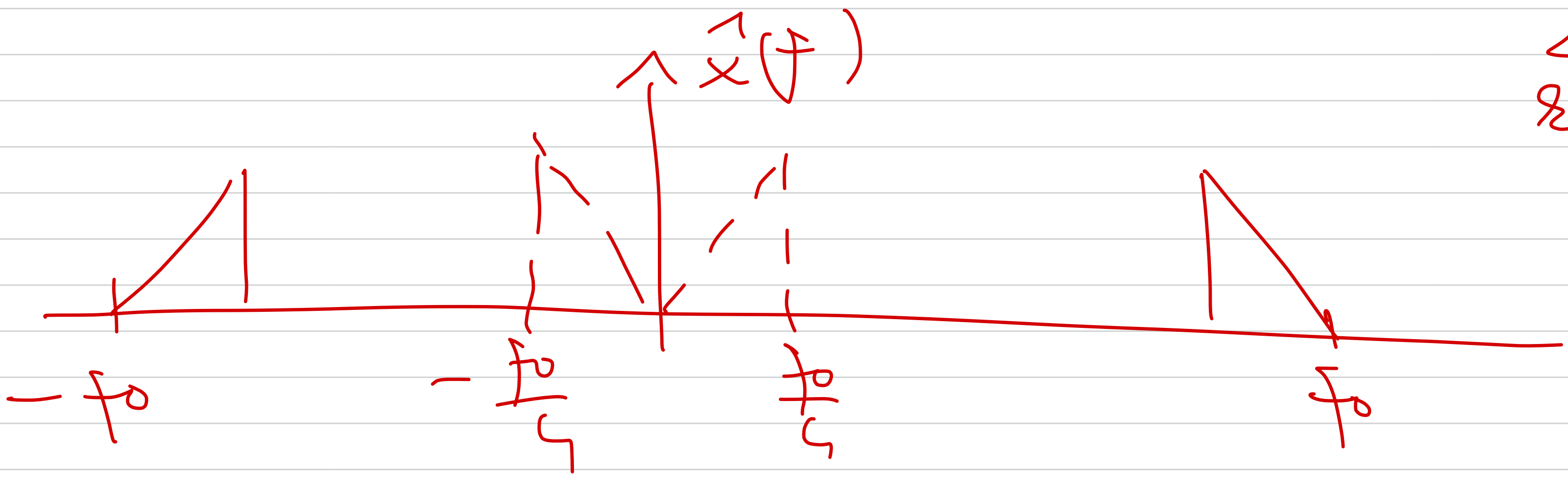
5.2. Sampling spectrally sparse signals

Assume that the spectrum has support in $[-f_0, f_0]$

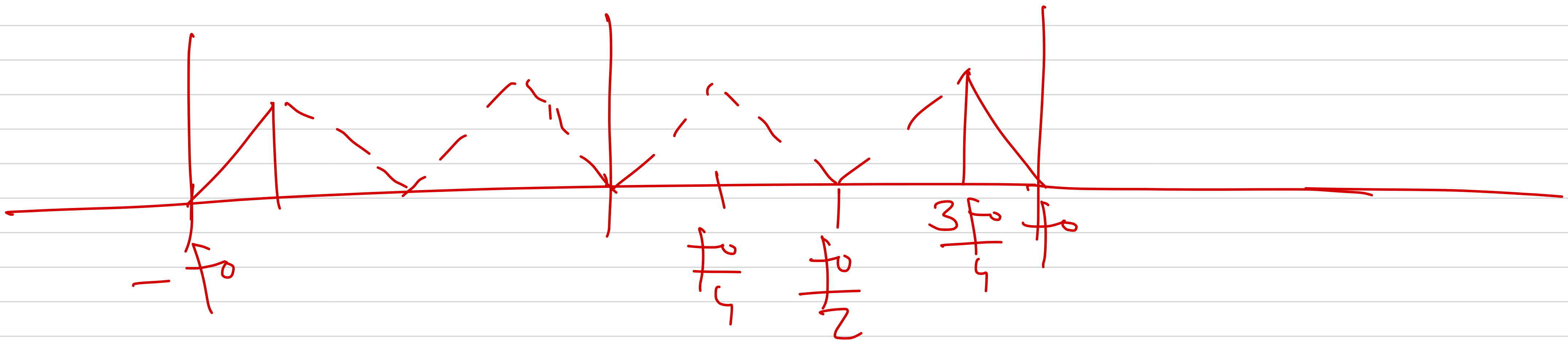


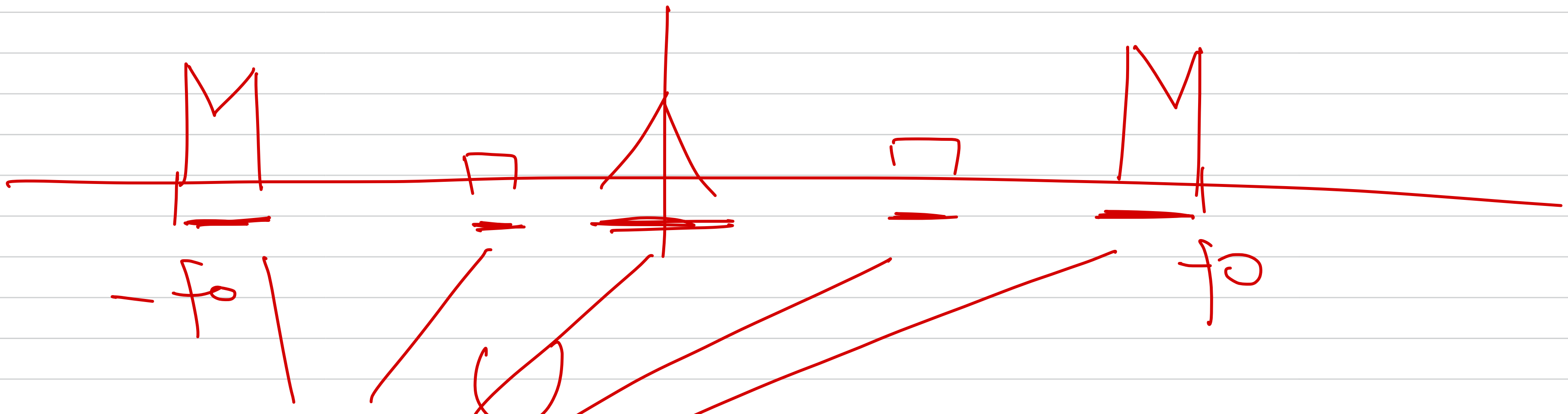
undersample by a factor of two, i.e., $f_s = f_0$, extract $\hat{x}(f)$

$$\sum_{k \in \mathbb{Z}} \hat{x}(f - k f_s)$$



undersample by a factor of 4, i.e., $f_s = \frac{f_0}{2}$





$\mathcal{I}$ = support set is $\ll 2f_0$

Case I: known support set $\leftarrow$ $m = s$

Case II: unknown support set $\leftarrow$ $m = 2s$

(finite-dim. case)

dual to sampling of signals with FRI

Assume the sampling set $P = \{t_n\}$, i.e., we are given the signal values $\{x(t_n)\}$.

Theorem 5.1. (Landau, 1967). To reconstruct x stably, we need

$$\mathcal{D}^-(P) = \lim_{r\to\infty} \inf_{t\in\mathbb{R}} \frac{|P \cap [t, t+r]|}{r} \geq |I|$$

where I is the spectral support set of x, i.e., the support set of $\hat{x}$, and $\mathcal{D}^-(P)$ is called the lower Beurling density.

$\mathcal{D}^-(P)$ is a generalization of sampling rate f_s. In the case of regular sampling, $\mathcal{D}^-(P)$ reduces to f_s (to be shown later).

Interpretation of $\mathcal{D}^-(P)$.

1. Fix r

2. Slide a window of length r across the time axis, i.e., $[t, t+r]$ and look for smallest no. of sampling points in any of these intervals and divide by r. This is a function of r, $f(r)$.

3. Take $r \to \infty$ and compute the limit of $f(r)$, $r\to\infty$.

regular sampling, we sample at integer multiples of $\frac{1}{f_s}$

$$f_s = \frac{1}{T} \Rightarrow T = \frac{1}{f_s}$$

window of length r contains $\frac{r}{\frac{1}{T}}$ sampling points

$$= r f_s \quad \text{—— } n \text{ ——}$$

$$\frac{r f_s}{r} = \boxed{f_s \geq |I|}$$

5.2.2. Stable sampling

Def. 5.2. A set of points $P = \{t_n\}$ is called a stable sampling set if for all $x_1, x_2 \in \mathcal{H}$, (set)

$$A \|x_1 - x_2\|_{\mathcal{H}}^2 \leq \|x_1(P) - x_2(P)\|_2^2 \leq B \|x_1 - x_2\|_{\mathcal{H}}^2, \quad A, B > 0$$

$A \leq B$

$$x_1(P) = \{x_1(t_1), x_1(t_2), \ldots, \ldots\} \quad \{t_i\} = P$$

if $\mathcal{H}$ is a vector space $\Rightarrow x_1 - x_2 \in \mathcal{H}$, if $x_1, x_2 \in \mathcal{H}$

$$\|Tx\|_2^2$$

$$A\|x\|_{\mathcal{H}}^2 \le \|x(P)\|_2^2 \le B\|x\|_{\mathcal{H}}^2$$

$$Tx$$

Sampling operator

known support set $\Rightarrow$ $\mathcal{H}$ is a vector space

unknown support set $\Rightarrow$ $\mathcal{H}$ is not a vector space