Mathematics of Information
The Fourier transform on $L^2(\mathbb{R})$.

These notes are based on [1, Chap. 1].

1 Failure of the Fourier integral for the sinc function

The Fourier transform on $\mathbb{R}$ is usually introduced as an operator mapping a function $f : \mathbb{R} \to \mathbb{C}$ to another function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by the integral

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} \, dx, \quad \forall \omega \in \mathbb{R},$$

without paying too much attention to the sense in which the integral on the right-hand side of (1) is defined. To see why we can run into trouble if we are not being careful, we consider the following example:

**Examples.** 1. Let $f(x) = \frac{\sin(\pi x)}{\pi x}$ be the normalized sinc function, whose Fourier transform is commonly known to be the rectangular function $1_{[-\frac{1}{2}, \frac{1}{2}]}(\omega)$. Suppose we wish to verify this at $\omega = 0$ by using the formula (1). In other words, we wish to evaluate

$$\hat{f}(0) = \int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{\pi x} \, dx.$$  

(2)

We split the domain of integration into sets $S_+ = \{ x \in \mathbb{R} : f(x) \geq 0 \}$ and $S_- = \{ x \in \mathbb{R} : f(x) < 0 \}$ and calculate

$$\int_{S_+} f(x) \, dx = \sum_{n=0}^{\infty} \int_{2n}^{2n+1} \frac{\sin(\pi x)}{\pi x} \, dx + \sum_{n=0}^{\infty} \int_{2n-1}^{2n} \frac{\sin(\pi x)}{\pi x} \, dx$$

$$\geq 2 \sum_{n=0}^{\infty} \int_{2n}^{2n+1} \frac{\sin(\pi x)}{\pi x} \, dx = 2 \sum_{n=0}^{\infty} \frac{2/\pi}{(2n+1)\pi} = \infty.$$  

Similarly, on $S_-$ we have

$$\int_{S_-} f(x) \, dx = 2 \sum_{n=0}^{\infty} \int_{2n+1}^{2n+2} \frac{\sin(\pi x)}{\pi x} \, dx$$

$$\leq 2 \sum_{n=0}^{\infty} \int_{2n+1}^{2n+2} \frac{\sin(\pi x)}{(2n+2)\pi} \, dx$$

$$= 2 \sum_{n=0}^{\infty} \frac{-2/\pi}{(2n+2)\pi} = -\infty,$$
We could now try to evaluate
\[\int_{-\infty}^{+\infty} f(x)dx = \int_{S_+} \sin(\pi x)\,dx + \int_{S_-} \sin(\pi x)\,dx = \infty - \infty, \] (3)
which is meaningless.

2. Let \(f : \mathbb{R} \to \mathbb{R}\) be an absolutely integrable function, that is \(f \in L^1(\mathbb{R})\). Define the sets \(S_+ = \{x \in \mathbb{R} : f(x) \geq 0\}\) and \(S_- = \{x \in \mathbb{R} : f(x) < 0\}\) as in the previous example. Then
\[\int_{S_+} f(x)dx = \int_{\mathbb{R}} 1_{S_+}(x)|f(x)|\,dx \leq \int_{\mathbb{R}} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R})} < \infty\]
and
\[\int_{S_-} f(x)dx = -\int_{\mathbb{R}} 1_{S_-}(x)|f(x)|\,dx \geq -\int_{\mathbb{R}} |f(x)|\,dx = -\|f\|_{L^1(\mathbb{R})} > -\infty,\]
so
\[\int_{\mathbb{R}} f(x)dx = \int_{S_+} f(x)dx + \int_{S_-} f(x)dx\] (4)
is a sum of finite numbers. We see that a pathology such as (3) cannot happen if \(f \in L^1(\mathbb{R})\).

The above examples suggest that it might be a bad idea to try to work directly with integrals with infinite limits such as (1) when \(f\) is not in \(L^1(\mathbb{R})\). Instead, we could interpret (1) as the limit
\[\hat{f}(\omega) = \lim_{R \to \infty} \int_{-R}^{R} f(x)e^{-2\pi i\omega x}\,dx = \lim_{R \to \infty} \int_{-\infty}^{+\infty} 1_{[-R,R]}(x)f(x)e^{-2\pi i\omega x}\,dx.\] (5)
Then, as we will see in the example below, if \(f \in L^2(\mathbb{R})\), then the function \(1_{[-R,R]}(x)f(x)e^{-2\pi i\omega x}\) is in \(L^1(\mathbb{R})\), for each fixed \(R\) and \(\omega\), and so its integral is well-behaved in the sense of (4). This is true even if \(f \notin L^1(\mathbb{R})\), e.g., the sinc function \(f(x) = \frac{\sin(\pi x)}{\pi x}\) is in \(L^2(\mathbb{R})\), but not in \(L^1(\mathbb{R})\). It can be verified for the sinc function that the limit (5) indeed exists for any fixed \(\omega\), and can be evaluated to be \(1_{[-\frac{1}{2},\frac{1}{2}]}(\omega)\) by using the calculus of residues.

Our goal now is to show how the Fourier transform can be defined for functions such as \(f(x) = \frac{\sin(\pi x)}{\pi x}\) for which the integral (1) is not well-defined. We will do this by applying the tools we developed in the first chapter to formalize the limit (5) for a general function \(f \in L^2(\mathbb{R})\).

Through this chapter we will write \(\hat{f}\) for the Fourier transform of a function \(f \in L^1(\mathbb{R})\) as given by the formula (1). The Fourier transform that we will define for functions that are in \(L^2(\mathbb{R})\) (but not necessarily in \(L^1(\mathbb{R})\)) will later be denoted by \(\mathcal{F}\), and we call this the \(L^2\)-Fourier transform. We will also see that the transforms \(\hat{\cdot}\) and \(\mathcal{F}\) of functions in \(L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) coincide. Before continuing, we give several more examples that illustrate the difference between the Banach space \(L^1(\mathbb{R})\) and the Hilbert space \(L^2(\mathbb{R})\).

**Examples.**
1. If \([-R,R]\) is a finite interval, then \(L^2([-R,R]) \subset L^1([-R,R])\), and for every \(f \in L^2([-R,R])\) we have \(\|f\|_{L^1([-R,R])} \leq \sqrt{2R} \|f\|_{L^2([-R,R])}\). Indeed, we can apply the Cauchy-Schwarz inequality in the Hilbert space \(L^2([-R,R])\)
to the functions \( f \) and \( \text{sgn}(f) \) to yield

\[
\| f \|_{L^1((-R,R))} = \left| \int_{-R}^{R} f(x) \text{sgn}(f(x)) \, dx \right| = |\langle f, \text{sgn}(f) \rangle |
\]

\[
\leq \| f \|_{L^2((-R,R))} \| \text{sgn}(f) \|_{L^2((-R,R))}
\]

\[
\leq \| f \|_{L^2((-R,R))} \left( \int_{-R}^{R} 1^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{2R} \| f \|_{L^2((-R,R))} < \infty
\]

2. If \( R = \infty \), then the argument in the previous example breaks down since we may have \( \| \text{sgn}(f) \|_{L^2(\mathbb{R})} = \infty \). Consider the function \( f(x) = \min\{1, \frac{1}{|x|}\} \). Then

\[
\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-1}^{1} 1^2 \, dx + \int_{-\infty}^{-1} \frac{1}{x^2} \, dx + \int_{1}^{\infty} \frac{1}{x^2} \, dx
\]

\[
= 2 + \frac{-1}{x} \bigg|_{-\infty}^{-1} + \frac{-1}{x} \bigg|_1^\infty = 4 < \infty,
\]

so \( f \in L^2(\mathbb{R}) \). However,

\[
\int_{-\infty}^{+\infty} |f(x)| \, dx = \int_{-1}^{1} 1 \, dx + \int_{-\infty}^{-1} \frac{1}{x} \, dx + \int_{1}^{\infty} \frac{1}{x} \, dx
\]

\[
= 2 + (- \log(-x)) \bigg|_{-\infty}^{-1} + \log(x) \bigg|_1^\infty = \infty,
\]

so \( f \notin L^1(\mathbb{R}) \). Therefore we have shown that \( L^2(\mathbb{R}) \nsubseteq L^1(\mathbb{R}) \). In fact, we can show similarly that the function \( f(x) = \mathbb{1}_{[0,1]}(x) \frac{1}{\sqrt{x}} \) is in \( L^1(\mathbb{R}) \) but not in \( L^2(\mathbb{R}) \), and so \( L^1(\mathbb{R}) \nsubseteq L^2(\mathbb{R}) \). In other words, neither of \( L^1(\mathbb{R}) \) is a subset of each other \( L^2(\mathbb{R}) \).

2 Translation, modulation, and the Fourier transform of the gaussian

For \( x, \xi \in \mathbb{R} \) we define the following linear operators on \( L^2(\mathbb{R}) \):

\[
(T_x f)(t) = f(t - x) \quad \forall t \in \mathbb{R},
\]

\[
(M_\xi f)(t) = e^{2\pi i \xi t} f(t) \quad \forall t \in \mathbb{R},
\]

where \( T_x \) is called the translation by \( x \), and \( M_\xi \) is called the modulation by \( \xi \).

**Example.**

For any \( x, \xi \in \mathbb{R} \) the maps \( T_x \) and \( M_\xi \) are linear maps on \( L^1(\mathbb{R}) \). Moreover, they are invertible with inverses \( T_{-x}^{-1} = T_{-x} \) and \( M_{-\xi}^{-1} = M_{-\xi} \). We can compute for an arbitrary \( f \in L^1(\mathbb{R}) \)

\[
\| T_x f \|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(t - x)| \, dt \leq \int_{-\infty}^{\infty} |f(t)| \, dt = \| f \|_{L^1(\mathbb{R})}
\]

\[
\| M_\xi f \|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |e^{2\pi i \xi t} f(t)| \, dt = \int_{-\infty}^{\infty} |f(t)| \, dt = \| f \|_{L^1(\mathbb{R})},
\]

\[
\| f \|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(t)| \, dt
\]

\[
\leq \int_{-\infty}^{\infty} \sqrt{2R} \| f \|_{L^2((-R,R))} \]
so $T_x$ and $M_\xi$ are linear isometries on $L^1(\mathbb{R})$. A similar calculation shows that they are also linear isometries on $L^2(\mathbb{R})$.

A gaussian is a function of the form $f(t) = ae^{-bt^2}$, where $a > 0$, $b > 0$ are fixed real numbers. We now show that the Fourier transform of a gaussian with $b = \pi$ is the gaussian itself:

**Lemma 1.** Let $\varphi(t) = e^{-\pi t^2}$ be the normalized Gaussian. The Fourier transform of $\varphi$ as given by (1) is the function $\varphi$ itself.

**Proof.** Note that $\varphi \in L^1(\mathbb{R})$ and $|\varphi(t)e^{-2\pi i\omega t}| \leq \varphi(t)$, so $\varphi(t)e^{-2\pi i\omega t}$ is in $L^1(\mathbb{R})$ for every fixed $\omega$, and thus $\varphi$ has a Fourier transform defined by the integral (1). What is more, the function $-2\pi ti\varphi(t)e^{-2\pi i\omega t}$ obtained by differentiating $\varphi(t)e^{-2\pi i\omega t}$ with respect to $\omega$ satisfies $|-2\pi ti\varphi(t)e^{-2\pi i\omega t}| \leq 2\pi t \varphi(t) \in L^1(\mathbb{R})$, and so we can integrate under the integral sign to obtain

$$\frac{d}{d\omega} \hat{\varphi}(\omega) = \frac{d}{d\omega} \int_{-\infty}^{+\infty} e^{-\pi t^2} e^{-2\pi i\omega t} dt = \int_{-\infty}^{+\infty} -2\pi i te^{-\pi t^2} e^{-2\pi i\omega t} dt$$

$$= \int_{-\infty}^{+\infty} ie^{-\pi t^2} e^{-2\pi i\omega t} dt = \int_{-\infty}^{+\infty} ie^{-\pi t^2} e^{-2\pi i\omega t} dt$$

$$= 0 - \int_{-\infty}^{+\infty} ie^{-\pi t^2} (-2\pi i\omega e^{-2\pi i\omega t}) dt = -2\pi \omega \hat{\varphi}(\omega),$$

for all $\omega \in \mathbb{R}$. From this it follows

$$\frac{d}{d\omega} (\hat{\varphi}(\omega)e^{\pi \omega^2}) = \left( \frac{d}{d\omega} (\hat{\varphi}(\omega)) + 2\pi i \omega \hat{\varphi}(\omega) \right) e^{\pi \omega^2} = 0 \cdot e^{\pi \omega^2} = 0,$$

for all $\omega \in \mathbb{R}$, so the function $\omega \mapsto \hat{\varphi}(\omega)e^{\pi \omega^2}$ is constant on $\mathbb{R}$. Therefore

$$\hat{\varphi}(\omega)e^{\pi \omega^2} = \hat{\varphi}(0)e^{\pi 0^2} = \int_{-\infty}^{+\infty} e^{-\pi t^2} dt = 1,$$

for all $\omega \in \mathbb{R}$, and so $\hat{\varphi}(\omega) = e^{-\pi \omega^2} = \varphi(\omega)$, as desired. \hfill $\square$

The following examples show that many operations involving $T_x$, $M_\xi$ and the Fourier transform applied to $\varphi$ can be carried out explicitly.

**Examples.** 1. Let $f \in L^1(\mathbb{R})$ and $(x, \xi) \in \mathbb{R}^2$. We compute $\widehat{M_\xi T_x f}$ in terms of $\hat{f}$:

$$\widehat{M_\xi T_x f} = \int_{-\infty}^{+\infty} e^{2\pi i \xi t} f(t - x) e^{-2\pi i\omega t} dt$$

$$= e^{-2\pi i(\omega - \xi)x} \int_{-\infty}^{+\infty} f(t) e^{-2\pi i(\omega - \xi)t} dt$$

$$= e^{2\pi i\xi x} e^{-2\pi i\omega x} \hat{f}(\omega - \xi)$$

$$= e^{2\pi i\xi x} (M_{-\xi} T_x \hat{f})(\omega),$$

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Now let $\hat{\xi}T_x f = e^{2\pi i \xi x} M_{-x} T_\xi f$.

2. Let $\varphi(x) = e^{-\pi x^2}$ be the normalized gaussian. For any given $(x, \xi), (u, \eta) \in \mathbb{R}^2$ we compute the inner product $\langle M_{\xi} T_x \varphi, M_{\eta} T_u \varphi \rangle$ in the space $L^2(\mathbb{R})$:

$$\langle M_{\xi} T_x \varphi, M_{\eta} T_u \varphi \rangle = \int_{-\infty}^{+\infty} e^{2\pi i \xi t} e^{-\pi (t-x)^2} e^{2\pi i \eta t} e^{-\pi (t-u)^2} dt$$

$$= \int_{-\infty}^{+\infty} e^{-\pi [(t-x)^2+(t-u)^2]} e^{-2\pi i (\eta-\xi) t} dt$$

$$= \int_{-\infty}^{+\infty} e^{-\pi \frac{1}{2} (2t-x-u)^2 + \frac{1}{2} (u-x)^2} e^{-2\pi i (\eta-\xi) t} dt$$

$$= \frac{1}{\sqrt{2}} e^{-\pi \frac{u-x}{2}} \int_{-\infty}^{+\infty} e^{-\pi y^2} e^{-2\pi i (\eta-\xi) y} e^{-\pi i (\eta-\xi) (x+u)} \frac{1}{\sqrt{2}} dy$$

$$= \frac{1}{\sqrt{2}} e^{-\pi \frac{u-x}{2}} e^{-\pi i (\eta-\xi) (x+u)} \hat{\varphi} \left( \frac{\eta - \xi}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} e^{-\pi i (\eta-\xi) (x+u)} \varphi \left( \frac{u - x}{\sqrt{2}} \right) \hat{\varphi} \left( \frac{\eta - \xi}{\sqrt{2}} \right)$$

3 Plancherel’s Theorem

We use the calculations in the previous examples to show the following Lemma which will be our first step towards defining the Fourier transform on $L^2(\mathbb{R})$.

**Lemma 2.** Let $\mathcal{X} = \text{span}\{M_{\xi} T_x \varphi : (x, \xi) \in \mathbb{R}^2\}$. Then we have the following

(i) $\mathcal{X}$ is a subspace of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

(ii) The Fourier transform $f \mapsto \hat{f}$ is a bijection from $\mathcal{X}$ to itself, and

(iii) $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, for all $f \in \mathcal{X}$.

**Proof.** (i) Note that, for any $(x, \xi) \in \mathbb{R}^2$, the function $M_{\xi} T_x \varphi$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, since $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $M_{\xi}$ are $T_x$ linear isometries on both of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Therefore any linear combination of such functions is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and so $\mathcal{X} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(ii) Now let $f = \sum_{k=1}^{n} c_k M_{\xi_k} T_{x_k} \varphi$ be an element of $\mathcal{X}$, where $c_k \in \mathbb{C}$ are complex coefficients. Then

$$\hat{f} = \sum_{k=1}^{n} c_k M_{\xi_k} T_{x_k} \varphi = \sum_{k=1}^{n} c_k e^{2\pi i \xi_k x_k} M_{-x_k} T_{\xi_k} \varphi \in \mathcal{X}.$$ 

This shows that the image of $\mathcal{X}$ under the Fourier transform lies in $\mathcal{X}$. To show surjectivity, we need to show that every $f \in \mathcal{X}$ is the Fourier transform of some element of $\mathcal{X}$. Let $f$ be as above and define $g = \sum_{k=1}^{n} c_k e^{2\pi i \xi_k x_k} M_{x_k} T_{-\xi_k} \varphi$. Then

$$\hat{g} = \sum_{k=1}^{n} c_k e^{2\pi i \xi_k x_k} \cdot e^{-2\pi i \xi_k x_k} M_{\xi_k} T_{x_k} \varphi = f,$$

as desired.
We first compute for any given \((x, \xi), (u, \eta) \in \mathbb{R}^2:\)

\[
\langle M_{\xi} T_x \varphi, M_{\eta} T_u \varphi \rangle = \langle e^{2\pi i \xi x} M_{-x} T_\xi \varphi, e^{2\pi i \eta u} M_{-u} T_\eta \varphi \rangle \\
= e^{2\pi i (\xi - \eta) x} \langle M_{-x} T_\xi \varphi, M_{-u} T_\eta \varphi \rangle \\
= e^{2\pi i (\xi - \eta) x} \frac{1}{\sqrt{2}} e^{-\pi i (-u + x)(\xi + \eta)} \varphi \left( \frac{\eta - \xi}{\sqrt{2}} \right) \varphi \left( \frac{-u + x}{\sqrt{2}} \right) \\
= \frac{1}{\sqrt{2}} e^{-\pi i (\eta - \xi)(x + u)} \varphi \left( \frac{u - x}{\sqrt{2}} \right) \varphi \left( \frac{\eta - \xi}{\sqrt{2}} \right) \\
= \langle M_\xi T_x \varphi, M_\eta T_u \varphi \rangle.
\]

Now let \(f \in \mathcal{X}\) be as above. Then

\[
\|f\|_{L^2(\mathbb{R})}^2 = \left\langle \sum_{k=1}^{n} c_k M_{\xi_k} T_{x_k} \varphi, \sum_{l=1}^{n} c_l M_{\xi_l} T_{x_l} \varphi \right\rangle \\
= \sum_{k, l=1}^{n} c_k \overline{c_l} \langle M_{\xi_k} T_{x_k} \varphi, M_{\xi_l} T_{x_l} \varphi \rangle \\
= \sum_{k, l=1}^{n} c_k \overline{c_l} \langle M_{\xi_k} T_{x_k} \varphi, M_{\xi_l} T_{x_l} \varphi \rangle \\
= \left\langle \sum_{k=1}^{n} c_k M_{\xi_k} T_{x_k} \varphi, \sum_{l=1}^{n} c_l M_{\xi_l} T_{x_l} \varphi \right\rangle = \|f\|_{L^2(\mathbb{R})}^2.
\]

\(\square\)

**Lemma 3.** Let \(\mathcal{X}\) be as in the previous lemma. Then \(\mathcal{X}\) is dense in \(L^2(\mathbb{R})\), that is \(\overline{\mathcal{X}} = L^2(\mathbb{R})\).

**Proof.** Fix an \(f \in L^2(\mathbb{R})\) and an \(\epsilon > 0\). Since \(\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty\), we have

\[
\|f - 1_{[-R, R]} f\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{+\infty} 1_{\mathbb{R} \setminus [-R, R]}(t) |f(t)|^2 dt \to 0,
\]
as \(R \to \infty\). Fix an \(R > 0\) large enough so that \(\|f - 1_{[-R, R]} f\|_{L^2(\mathbb{R})} < \epsilon/2\). Also fix a large enough \(A > 0\) so that \(A > R\) and

\[
\frac{2e^{-2\pi A^2}}{1 - e^{-2\pi A^2}} \left( e^{\pi R^2} \|f\|_{L^2(\mathbb{R})} + \epsilon \right)^2 < \frac{\epsilon^2}{8}.
\]

(6)

The significance of this choice will become apparent later. Now consider the function \(g(t) = 1_{[-R, R]}(t) f(t) e^{\pi t^2}\). Note that

\[
\int_{-A}^{A} |g(t)|^2 dt = \int_{-A}^{A} 1_{[-R, R]}(t) |f(t)|^2 e^{2\pi t^2} dt \leq e^{2\pi R^2} \int_{-A}^{A} |f(t)|^2 dt \leq e^{2\pi R^2} \|f\|_{L^2(\mathbb{R})}^2 < \infty,
\]

(7)

so \(g \in L^2([-A, A])\). Thus the function \(g\) can be approximated by its partial Fourier series on \([-A, A]\), that is, there exists an \(N \in \mathbb{N}\) and coefficients \((c_k)_{k=-N}^{N}\) such that

\[
s(t) = \sum_{k=-N}^{N} c_k e^{\pi i k t / A}
\]
satisfies \( \|g - s\|^2_{L^2([-A,A])} < \frac{\epsilon^2}{8} \). We now bound \( \|s\|_{L^2([-A,A])} \) by using (7) as follows:

\[
\|s\|^2_{L^2([-A,A])} \leq \|g\|^2_{L^2([-A,A])} + \|s - g\|^2_{L^2([-A,A])} \leq e^{\pi R^2} \|f\|^2_{L^2(\mathbb{R})} + \sqrt{\frac{\epsilon^2}{8}} < e^{\pi R^2} \|f\|^2_{L^2(\mathbb{R})} + \epsilon.
\]

Now define the function \( h \) by

\[
h(t) := s(t)e^{-\pi t^2} = \sum_{k=-N}^{N} c_k e^{\pi i k t / A} \varphi(t) = \sum_{k=-N}^{N} c_k M_k \varphi(t),
\]

so that we have \( h \in \mathcal{X} \). By using the \( 2A \)-periodicity of \( s \) and fact that \( \mathbb{1}_{[-R,R]}f \) is zero outside the interval \([-A,A]\) we can estimate

\[
\|\mathbb{1}_{[-R,R]}f - h\|^2_{L^2(\mathbb{R})} = \sum_{n \in \mathbb{Z}} \int_{(2n-1)A}^{(2n+1)A} \| \mathbb{1}_{[-R,R]}(t)f(t) - h(t) \|^2 dt
\]

\[
= \int_{-A}^{A} \| \mathbb{1}_{[-R,R]}(t)f(t) - s(t)e^{-\pi t^2} \|^2 dt + \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{(2n-1)A}^{(2n+1)A} \|s(t)e^{-\pi t^2} \|^2 dt
\]

\[
\leq \int_{-A}^{A} |g(t) - s(t)|^2 e^{-2\pi t^2} dt + \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{(2n-1)A}^{(2n+1)A} |s(t)|^2 e^{-2\pi(2|n|-1)^2 A^2} dt
\]

\[
\leq \|g - s\|^2_{L^2([-A,A])} + \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-2\pi|n|A^2} \int_{-A}^{A} |s(t + 2nA)|^2 dt
\]

\[
= \|g - s\|^2_{L^2([-A,A])} + 2 \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-2\pi|n|A^2} \|s\|^2_{L^2([-A,A])}
\]

\[
< \frac{\epsilon^2}{8} + 2 \frac{e^{-2\pi A^2}}{1 - e^{-2\pi A^2}} \left( e^{\pi R^2} \|f\|^2_{L^2(\mathbb{R})} + \epsilon \right)^2
\]

\[
< \frac{\epsilon^2}{8} + \frac{\epsilon^2}{8} = \frac{\epsilon^2}{4},
\]

where the last line is due to (6). Therefore

\[
\|f - h\|^2_{L^2(\mathbb{R})} \leq \|f - \mathbb{1}_{[-R,R]}f\|^2_{L^2(\mathbb{R})} + \|\mathbb{1}_{[-R,R]}f - h\|^2_{L^2(\mathbb{R})} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

We have shown that an arbitrary element \( f \) of \( L^2(\mathbb{R}) \) can be approximated to arbitrary precision \( \epsilon \) by an element \( h \in \mathcal{X} \), and so \( \mathcal{X} \) is dense in \( L^2(\mathbb{R}) \). \( \square \)

With the previous two Lemmas we are finally in the position to define the Fourier transform on \( L^2(\mathbb{R}) \):

**Theorem 1.** Let \( \mathcal{X} \) be as in Lemma 2. Then the Fourier transform \( f \mapsto \hat{f} \) on \( \mathcal{X} \) extends to a unitary operator \( \mathcal{F} \) on \( L^2(\mathbb{R}) \).

**Proof.** Since \( \mathcal{X} \subset L^1(\mathbb{R}) \), we can define \( \mathcal{F}f := \hat{f} \), for \( f \in \mathcal{X} \). Now let \( f \) be an arbitrary element of \( L^2(\mathbb{R}) \). By Lemma 3 there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathcal{X} \) such that \( \|f_n - f\|_{L^2(\mathbb{R})} \to 0 \) as \( n \to \infty \). Since the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges, we know it is a Cauchy sequence. Now, by Lemma 2 (iii) we have \( \|\hat{f}_n - \hat{f}_m\|_{L^2(\mathbb{R})} = \|f_n - f_m\|_{L^2(\mathbb{R})} \) for \( m, n \in \mathbb{N} \), so \( \{\hat{f}_n\}_{n \in \mathbb{N}} \) is also a Cauchy sequence. Since the space \( L^2(\mathbb{R}) \) is complete, it follows that \( \{\hat{f}_n\}_{n \in \mathbb{N}} \) converges to some element of \( L^2(\mathbb{R}) \).
We set $Ff = \lim_{n \to \infty} \hat{f}_n$. As $\hat{\cdot}$ and $\lim$ are linear, it follows immediately that $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined in this way is linear. Moreover,

$$\|Ff\|_{L^2(\mathbb{R})} = \|\lim_{n \to \infty} \hat{f}_n\|_{L^2(\mathbb{R})} = \lim_{n \to \infty} \|\hat{f}_n\|_{L^2(\mathbb{R})} = \lim_{n \to \infty} \|f_n\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})},$$

so $F$ is an isometry. In order to conclude that $F$ is unitary, it remains to show that $F$ is surjective. Fix an arbitrary element $f$ of $L^2(\mathbb{R})$ and let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{X}$ such that $\lim_{n \to \infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0$ as before. By Lemma 2 (ii) we can find another sequence $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{X}$ such that $f_n = \hat{g}_n$, for all $n \in \mathbb{N}$. Then, by Lemma 2 (iii) we have

$$\|g_n - g_m\|_{L^2(\mathbb{R})} = \|\hat{g}_n - \hat{g}_m\|_{L^2(\mathbb{R})} = \|f_n - f_m\|_{L^2(\mathbb{R})},$$

so $\{g_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, and so it converges to some $g \in L^2(\mathbb{R})$. Then, by definition of $F$,

$$Fg = \lim_{n \to \infty} \hat{g}_n = \lim_{n \to \infty} f_n = f,$$

where the limits are in $L^2(\mathbb{R})$. We have shown that an arbitrary $f \in L^2(\mathbb{R})$ is in the image of $F$, and so $F$ is surjective. Thus $F$ is a surjective linear isometry on a Hilbert space, and therefore it is unitary. $\square$

With some extra work, it can be shown that the space $\mathcal{X}$ in the statement of the previous theorem can be replaced with the whole of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, i.e., we have the following theorem.

**Theorem 2** (Plancherel). The Fourier transform $f \mapsto \hat{f}$ on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends to a unitary operator $F$ on $L^2(\mathbb{R})$.

Thus the Fourier transform of functions $f$ which are in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$ (such as $\frac{\sin(\pi x)}{\pi x}$) can be obtained as the limit of the $L^1$-Fourier transforms (as given by the formula (1)) of any functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that converge to $f$. In practice, this means that the Fourier transform can often be calculated just like we did heuristically in (5):

$$Ff(\omega) = \lim_{R \to \infty} F(\mathbb{1}_{[-R, R]} f)(\omega) = \lim_{R \to \infty} (\mathbb{1}_{[-R, R]} \hat{f})(\omega) = \lim_{R \to \infty} \int_{-\infty}^{+\infty} (\mathbb{1}_{[-R, R]} f)(x) e^{-2\pi i \omega x} dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) e^{-2\pi i \omega x} dx,$$

as $\mathbb{1}_{[-R, R]} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $R > 0$ and $\|f - \mathbb{1}_{[-R, R]} f\|_{L^2(\mathbb{R})} \to 0$ as $R \to \infty$.

**References**


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$^1$The limit is always valid, but in some cases it may be difficult to compute its exact value.