Examination on Mathematics of Information
August 29, 2018

Please note:

- Duration of exam: 180 minutes
- Maximum number of points: 100
- You are allowed to use any printed or handwritten material (i.e., books, lecture and discussion session notes, summaries), but no computers, tablets, smart phones or other electronic devices.
- Your solutions should be explained in detail and your handwriting needs to be clean and readable.
- Please do not use red or green pens. You may use pencils.
- Please note that the ETHZ “Disziplinarordnung RSETHZ 361.1” applies.

Before you start:
1. The problem statements consist of 7 pages. Please verify that you have received all 7 pages.
2. Please fill in your name and your Legi-number below.
3. Please place an identification document on your desk so we can verify your identity.

During the exam:
4. For your solutions, please use only the empty sheets provided by us. Should you need more paper, please let us know.

After the exam:
5. Please number all the sheets you want to turn in. Please specify below the number of additional sheets you want to turn in (excluding the sheets with the problem statements). All sheets containing problem statements must be turned in.

Surname: .......................... Given name: ..........................

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Signature: ........................................

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1. Problem 1

The Hilbert space $L^2(\mathbb{R}^2)$ consists of all $F : \mathbb{R}^2 \to \mathbb{C}$ such that $\|F\|_{L^2(\mathbb{R}^2)} < \infty$, where

$$\|F\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |F(x, y)|^2 \, dx \, dy = \int_{\mathbb{R}^2} |F(x, y)|^2 \, dy \, dx.$$

(a) Write down the definition of a unitary operator on a general Hilbert space.

(b) Let $f, g \in L^2(\mathbb{R})$. We define the short-time Fourier transform $V_g f : \mathbb{R}^2 \to \mathbb{C}$ of $f$ with respect to window $g$ by

$$(V_g f)(x, \omega) = \int_{\mathbb{R}} f(t) g(t - x) e^{-2\pi i \omega t} \, dt.$$ 

Consider the following two transformations:

(i) The asymmetric coordinate transform $T_a$ is defined for a function $F : \mathbb{R}^2 \to \mathbb{C}$ by

$$T_a F(x, y) = F(y, y - x).$$

Show that $T_a$ is a unitary operator on $L^2(\mathbb{R}^2)$.

[Hint: First show that $T_a$ maps $L^2(\mathbb{R}^2)$ functions to $L^2(\mathbb{R}^2)$ functions, and then compute the adjoint $T_a^*$ explicitly.]

(ii) The partial Fourier transform $F_2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the unique bounded linear operator on $L^2(\mathbb{R}^2)$ with the following property:

Whenever $F \in L^2(\mathbb{R}^2)$ is such that $F(x, \cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$, meaning that

$$\int_{\mathbb{R}} |F(x, y)| \, dy < \infty \quad \text{for all} \quad x \in \mathbb{R},$$

$F_2 F$ is given by the formula

$$(F_2 F)(x, \omega) = \int_{\mathbb{R}} F(x, t) e^{-2\pi i \omega t} \, dt, \quad \text{for all} \quad (x, \omega) \in \mathbb{R}^2. \quad (1)$$

You may use—without proof—the fact that $F_2$ is a unitary operator.

Show carefully that

$$V_g f = F_2 T_a (f \otimes g) \quad \text{for all} \quad f, g \in L^2(\mathbb{R}), \quad (2)$$

where for two functions $h_1, h_2 : \mathbb{R} \to \mathbb{C}$ we write $h_1 \otimes h_2$ to denote the function $(h_1 \otimes h_2)(x, y) = h_1(x) h_2(y)$.

[Please note that if you want to use (1) to compute $F_2 F$ for a function $F \in L^2(\mathbb{R}^2)$, you first have to verify that $F$ satisfies the additional assumption that $F(x, \cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$.]

(c) Using (2) deduce that $V_g f \in L^2(\mathbb{R}^2)$ for all $f, g \in L^2(\mathbb{R})$, and that

$$\langle V_g f_1, V_g f_2 \rangle = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle, \quad \text{for all} \quad f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}). \quad (3)$$
Problem 2

A multiresolution approximation is an increasing sequence \( \{V_j\}_{j \in \mathbb{Z}} \) of closed linear subspaces of \( L^2(\mathbb{R}) \) such that

(I) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \) and \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{R}) \),

(II) for all \( f \in L^2(\mathbb{R}) \) and \( j \in \mathbb{Z} \), \( f \in V_j \iff f(2 \cdot) \in V_{j+1} \),

(III) for all \( f \in L^2(\mathbb{R}) \) and \( k \in \mathbb{Z} \), \( f \in V_0 \iff f(\cdot - k) \in V_0 \),

(IV) there exists a function \( \varphi \in V_0 \) such that \( \{\varphi(\cdot - k) : k \in \mathbb{Z}\} \) is a Riesz basis of the space \( V_0 \).

Let \( \varphi \in L^2(\mathbb{R}) \) be given in the Fourier domain by

\[
\hat{\varphi}(\xi) = \begin{cases} 
1, & |\xi| \leq \frac{1}{3} \\
\cos\left(\frac{\pi}{2} \nu(3|\xi| - 1)\right), & \frac{1}{3} < |\xi| \leq \frac{2}{3} \\
0, & \text{otherwise}
\end{cases}
\]

where

\[
\nu(x) = \begin{cases} 
0, & x < 0 \\
x, & 0 \leq x < 1 \\
1, & 1 \leq x
\end{cases}
\]

(a) (i) Sketch \( \hat{\varphi} \) on the interval \([-1, 1]\).
(ii) Show that \( \{\varphi(\cdot - k) : k \in \mathbb{Z}\} \) is an orthonormal system, that is

\[
\langle \varphi(\cdot - k), \varphi(\cdot - l) \rangle = \begin{cases} 
0, & k \neq l \\
1, & k = l
\end{cases} \quad \text{for all } k, l \in \mathbb{Z}.
\]

You may use — without proof — the fact that, for any given \( g \in L^2(\mathbb{R}) \), we have that \( \{g(\cdot - k) : k \in \mathbb{Z}\} \) is an orthonormal system if and only if \( \sum_{n \in \mathbb{Z}} |\hat{g}(\xi + n)|^2 = 1 \), for all \( \xi \in \mathbb{R} \).

(b) Define \( V_0 \) to be the closure of \( \text{span}\{\varphi(\cdot - k) : k \in \mathbb{Z}\} \), and let \( V_j = \{f(2^j \cdot) : f \in V_0\} \) for \( j \in \mathbb{Z} \).

(i) Let \( P_{V_j} \) denote the orthogonal projection onto \( V_j \). You may use — without proof — that \( \{\varphi_{j,k} := 2^{\frac{j}{2}} \varphi(2^j \cdot - k) : k \in \mathbb{Z}\} \) is an orthonormal basis for \( V_j \), and that \( \|P_{V_j} f\|_{L^2(\mathbb{R})} \) is given by the following expression:

\[
\|P_{V_j} f\|_{L^2(\mathbb{R})}^2 = \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x) \overline{\varphi_{j,k}(x)} dx \right|^2, \quad \text{for all } f \in L^2(\mathbb{R}).
\]  

(4)

By applying the Plancherel identity to (4) show that

\[
\|P_{V_j} f\|_{L^2(\mathbb{R})}^2 + \|P_{V_j}(f(\cdot - 2^{-j+1}))\|_{L^2(\mathbb{R})}^2 = 2\|\hat{\varphi}(2^{-j} \cdot)\|_{L^2(\mathbb{R})}^2,
\]

(5)

for all \( j \in \mathbb{Z} \) and \( f \in L^2(\mathbb{R}) \).

[Hint: The Plancherel identity states that for any two functions \( h_1, h_2 \in L^2(\mathbb{R}) \),

\[
\int_{\mathbb{R}} h_1(x) \overline{h_2(x)} dx = \int_{\mathbb{R}} \hat{h_1}(\xi) \overline{\hat{h_2}(\xi)} d\xi.
\]  

\]
we have \(<h_1, h_2> = \langle \hat{h}_1, \hat{h}_2 >\). After applying this identity, combine the expression you obtain into one sum. Then reinterpret the resulting expression as an expansion in
\[ E = \{ e_m(\xi) = 2^{-\frac{j+1}{2}} e^{-\frac{2\pi im}{2^j+1}} : m \in \mathbb{Z} \}. \]
For this you will need to use the fact that \(\hat{\varphi}\) is zero outside a finite interval to replace the limits of the integrals accordingly. You may use — without proof—that \(E\) is an orthonormal basis for \(L^2([-2^j, 2^j])\).

(ii) For the remainder of the question you may use—without proof—that
\[ \lim_{j \to \infty} \| \hat{f} - \hat{\varphi}(2^{-j} \cdot) \hat{f} \|_{L^2(\mathbb{R})} = 0, \] and
\[ \lim_{j \to -\infty} \| \hat{\varphi}(2^{-j} \cdot) \hat{f} \|_{L^2(\mathbb{R})} = 0, \]
for any \(f \in L^2(\mathbb{R})\). Use these facts together with (5) to show that
\[ \lim_{j \to \infty} \| P_{V_j} f \|_{L^2(\mathbb{R})} = \| f \|_{L^2(\mathbb{R})}. \]
Deduce that \(\lim_{j \to \infty} \| f - P_{V_j} f \|_{L^2(\mathbb{R})} = 0\).

(iii) Use (5) to show that \(\lim_{j \to -\infty} \| P_{V_j} f \|_{L^2(\mathbb{R})} = 0\).

(c) Use the results you have obtained so far to prove that \(\{ V_j \}_{j \in \mathbb{Z}}\) defined in (b) is a multiresolution approximation of \(L^2(\mathbb{R})\).

[Hint: Recall the following facts, which you may use without proof, about orthonormal projections: \(P_{V_j}^2 = P_{V_j}\), \(P_{V_j}^* = P_{V_j}\), and \(\| P_{V_j} g \|_{L^2(\mathbb{R})} \leq \| g \|_{L^2(\mathbb{R})}\), for all \(g \in L^2(\mathbb{R})\).]

[Hint: Use (b)(ii) and (b)(iii) to prove item (I) in the definition of a multiresolution approximation. To show that \(\bigcup_{j \in \mathbb{Z}} V_j\) is dense in \(L^2(\mathbb{R})\), it suffices to establish that for any \(\epsilon > 0\), there exist a \(j \in \mathbb{Z}\) and a function \(\tilde{f} \in V_j\), such that \(\| f - \tilde{f} \|_{L^2(\mathbb{R})} < \epsilon\). To show this you can use \(\lim_{j \to \infty} \| f - P_{V_j} f \|_{L^2(\mathbb{R})} = 0\).]
Problem 3

For this problem we use the two-indices notation $f_{x,y}$ to denote time-frequency shifts, that is, if $f : \mathbb{R} \to \mathbb{C}$ is a function and $x, y \in \mathbb{R}$, then we shall write $f_{x,y}(t) = e^{2\pi i y t} f(t - x)$.

You may use—without proof—that for all $f, h \in L^2(\mathbb{R})$ the function $(x, y) \mapsto \langle f, h_{x,y} \rangle$ is an $L^2(\mathbb{R}^2)$ function, that is
\[
\int_{\mathbb{R}^2} |\langle f, h_{x,y} \rangle|^2 \, dx \, dy < \infty.
\]
Furthermore, you may also use—without proof—the following identity:
\[
\int_{\mathbb{R}^2} \langle f, g_{x,y} \rangle \overline{\langle u, v_{x,y} \rangle} \, dx \, dy = \langle f, u \rangle \overline{\langle g, v \rangle} \quad \text{for all } f, g, u, v \in L^2(\mathbb{R}). \tag{IR}
\]

(a) Consider a Weyl-Heisenberg system $G = \{g_{mT,nF}\}_{m,n \in \mathbb{Z}}$ with time-frequency parameters $T > 0$ and $F > 0$. Assume that $G$ is a frame for $L^2(\mathbb{R})$. Let $S$ be the corresponding frame operator and $\tilde{g} = S^{-1} g$ the canonical dual function. We know that $\tilde{G} = \{\tilde{g}_{mT,nF}\}_{m,n \in \mathbb{Z}}$ is the canonical dual frame to $G$, and that the following reconstruction formula holds:
\[
f = \sum_{m,n \in \mathbb{Z}} \langle f, \tilde{g}_{mT,nF} \rangle g_{mT,nF} \quad \text{for all } f \in L^2(\mathbb{R}).
\]
Using this reconstruction formula, prove that
\[
\langle f, h \rangle = \sum_{m,n \in \mathbb{Z}} \langle f, \tilde{g}_{mT,x,nF+y} \rangle \langle g_{mT,x,nF+y}, h \rangle \quad \text{for all } f, h \in L^2(\mathbb{R}) \text{ and all } x, y \in \mathbb{R}.
\]
[Hint: Expand $f_{-x,-y}$ using the reconstruction formula, and then take the inner product of both sides with $h_{-x,-y}$.]

(b) By integrating both sides of (6) over $(x, y) \in [0, T) \times [0, F)$ for a fixed pair of functions $f$ and $h$, show that $\langle g, \tilde{g} \rangle = TF$. Justify the validity of any manipulations you do by verifying the following absolute convergence property:
\[
\sum_{m,n \in \mathbb{Z}} \int_0^T \int_0^F |\langle f, \tilde{g}_{mT+x,nF+y} \rangle \overline{\langle g_{mT+x,nF+y}, h \rangle}| \, dx \, dy < \infty.
\]
[Hint: For both calculations, rewrite the resulting expression as an integral over $\mathbb{R}^2$ which does not involve infinite summation.]

(c) Assume the following result from the lectures without proof:

**Lemma.** Denote by $K$ a countable index set. Let $\{h_k\}_{k \in K}$ be a frame for a Hilbert space $\mathcal{H}$ and $\{\tilde{h}_k\}_{k \in K}$ its canonical dual frame. For a fixed $z \in \mathcal{H}$, let $c_k = \langle z, \tilde{h}_k \rangle$ so that
\[ z = \sum_{k \in \mathcal{K}} c_k h_k. \] If it is possible to find scalars \( \{a_k\}_{k \in \mathcal{K}} \) such that \( z = \sum_{k \in \mathcal{K}} a_k h_k \), then we must have
\[ \sum_{k \in \mathcal{K}} |a_k|^2 = \sum_{k \in \mathcal{K}} |c_k|^2 + \sum_{k \in \mathcal{K}} |c_k - a_k|^2. \]

Find two distinct sets \( \{a_{m,n}\}_{m,n \in \mathbb{Z}} \) such that \( g = \sum_{m,n \in \mathcal{K}} a_{m,n} g_{mT,nF} \), and then use the Lemma to deduce that \( TF \leq 1 \).
Problem 4

Define the following local-averaging operator

\[(Ax)_n = \int_{n-1/2}^{n+1/2} x(t)dt, \quad n \in \mathbb{Z},\]

that takes in a function \(x \in L^2(\mathbb{R})\) and yields a sequence \(\{(Ax)_n\}_{n \in \mathbb{Z}}\) of local averages.

(a) Verify that \(A\) is a bounded linear operator from \(L^2(\mathbb{R})\) to \(\ell^2(\mathbb{Z})\) and compute the adjoint \(A^* : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})\) of \(A\).

(b) Show that \(\|A^*y\|_{L^2(\mathbb{R})} = \|y\|_{\ell^2(\mathbb{Z})}\) for all \(y \in \ell^2(\mathbb{Z})\).

(c) Define \(\text{Im}(A^*) = \{A^*y : y \in \ell^2(\mathbb{Z})\}\). You may use — without proof — that \(\text{Im}(A^*)\) is a closed subspace of \(L^2(\mathbb{R})\), and thus a Hilbert space in its own right. For each \(n \in \mathbb{Z}\) let \(e_n = \mathbb{1}_{[n-1/2,n+1/2]}\) be the indicator function of the interval \([n - 1/2, n + 1/2]\). Show that \(G := \{e_n : n \in \mathbb{Z}\}\) is a subset of \(\text{Im}(A^*)\), and that \(G\) is a frame for \(\text{Im}(A^*)\). Show that \(A\) can be interpreted as the analysis operator associated with the frame \(G\).