Problem 1  Overcomplete expansion in \( \mathbb{R}^2 \)

Consider the following example discussed in class. For the vectors
\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad e_3 = e_1 - e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]
we found that any vector \( x \in \mathbb{R}^2 \) can be represented according to
\[
x = \langle x, \hat{e}_1 \rangle e_1 + \langle x, \hat{e}_2 \rangle e_2 + \langle x, \hat{e}_3 \rangle e_3
\]
where
\[
\hat{e}_1 = 2e_1, \quad \hat{e}_2 = -e_3, \quad \hat{e}_3 = -e_1.
\]

a) Find another set of vectors \( \tilde{e}_1', \tilde{e}_2', \tilde{e}_3' \), neither of which is collinear to neither of the vectors \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) and such that any vector \( x \in \mathbb{R}^2 \) can be represented as
\[
x = \langle x, \tilde{e}_1' \rangle e_1 + \langle x, \tilde{e}_2' \rangle e_2 + \langle x, \tilde{e}_3' \rangle e_3.
\]

Hint: Look for another right-inverse of the matrix
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1
\end{bmatrix}.
\]

b) Now consider the following example discussed in class. For the vectors
\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \hat{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}
\]
any vector \( x \in \mathbb{R}^2 \) can be represented as
\[
x = \langle x, \hat{e}_1 \rangle \hat{e}_1 + \langle x, \hat{e}_2 \rangle \hat{e}_2.
\]

Show that \( x \) can also be written as
\[
x = \langle x, \hat{e}_1 \rangle e_1 + \langle x, \hat{e}_2 \rangle e_2.
\]

Is it possible to find two vectors \( e_1', e_2' \), neither of which is collinear to neither of the vectors \( e_1, e_2 \) such that
\[
x = \langle x, e_1' \rangle \hat{e}_1 + \langle x, e_2' \rangle \hat{e}_2?
\]
If the answer is “yes”, find these vectors. If the answer is “no”, explain why this is not possible.

**Problem 2  Equality in the Cauchy-Schwarz inequality**

Prove that if the elements \( x \) and \( y \) of a complex Hilbert space \( \mathcal{H} \) satisfy \( |\langle x, y \rangle| = \|x\|\|y\| \) and \( y \neq 0 \), then \( x = cy \) for some \( c \in \mathbb{C} \).

*Hint:* Assume \( \|x\| = \|y\| = 1 \) and \( \langle x, y \rangle = 1 \). Then \( x - y \) and \( x \) are orthogonal, while \( x = x - y + y \). Therefore, \( \|x\|^2 = \|x - y\|^2 + \|y\|^2 \).

**Problem 3  Parallelogram law**

a) Let \( (\mathcal{X}, \langle \cdot, \cdot \rangle) \) be an inner product space and \( \| \cdot \| \) the norm induced by \( \langle \cdot, \cdot \rangle \). Show that for all \( x, y \in \mathcal{X} \), the following holds

\[
\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2. 
\]

(1)

b) Let \( (\mathcal{X}, \| \cdot \|) \) be a normed space. Show that if (1) holds for all \( x, y \in \mathcal{X} \), then there exists an inner product \( \langle \cdot, \cdot \rangle \) such that \( \|x\| = \sqrt{\langle x, x \rangle} \) for all \( x \in \mathcal{X} \). For simplicity, you may assume that \( \mathcal{X} \) is a real normed space.

Use the last statement to show that \( \ell^2(\mathbb{Z}) \) is a Hilbert space. What about \( \ell^1(\mathbb{Z}) \)?

*Recall:* \( \ell^p(\mathbb{Z}) \), \( p \in [1, \infty) \), is the space

\[
\ell^p(\mathbb{Z}) \triangleq \left\{ \{u_k\}_{k \in \mathbb{Z}} : \sum_{k=\infty}^{+\infty} |u_k|^p < \infty \right\}
\]

equipped with the norm

\[
\|u\|_{\ell^p(\mathbb{Z})} \triangleq \left( \sum_{k=\infty}^{+\infty} |u_k|^p \right)^{1/p}.
\]

**Problem 4  Projection on closed subspaces**

Let \( \mathcal{H} \) be a Hilbert space, \( x \in \mathcal{H} \), and \( S \) a closed subspace of \( \mathcal{H} \). We know that there exists a \( y \in S \) such that

\[
\|x - y\| = \min_{z \in S} \|x - z\|. 
\]

(2)

a) Use the parallelogram law to show that \( y \) is the unique element in \( S \) fulfilling (2).

b) Show that \( y \) is the unique element in \( S \) such that \( (x - y) \in S^\perp \).
Problem 5  A surjective linear isometry is unitary
Let $\mathcal{H}$ be a real Hilbert space and $T: \mathcal{H} \to \mathcal{H}$ a surjective linear isometry. By applying the polarization formula
\[ \langle x, y \rangle = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right), \quad \forall x, y \in \mathcal{H}, \]
show that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. Deduce that $T$ is unitary. Where did you use the fact that $T$ is surjective?

Problem 6  Unconditional convergence
Let $\mathcal{H} = l_2(\mathbb{N})$ and define
\[ x_k = (0, \ldots, 0, 1/k, 0, \ldots, 0, \ldots), \quad k \in \mathbb{N}, \]
where the only non-zero entry $1/k$ of the sequence $x_k$ is at position $k \in \mathbb{N}$. Does the sum
\[ \sum_{k=1}^{\infty} x_k \]
converge unconditionallly?

Problem 7  Discrete Fourier Transform (DFT) as a signal expansion
The DFT of an $N$-point signal $f(n)$, $n = 0, 1, \ldots, N - 1$, is defined as
\[ \tilde{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{k}{N} n} \]
Find the corresponding inverse transform and show that the DFT can be interpreted as a signal expansion in $\mathbb{C}^N$. 