Problem 1  Change of basis matrix between ONBs is unitary
Let $B_1 = \{g_n\}_{n=1}^N$ and $B_2 = \{h_n\}_{n=1}^N$ be orthonormal bases for the Hilbert space $\mathbb{C}^N$. Show that the matrix $U \in \mathbb{C}^{N \times N}$ whose entries are given by $U_{jk} = \langle h_k, g_j \rangle_{\mathbb{C}^N}$ is unitary.

Problem 2  Oversampled A/D conversion
Let $T, B > 0$ be such that $1/T > 2B$, and set $g_k(t) = 2Bsinc(2B(t - kT))$, for $k \in \mathbb{Z}$. We have seen in the lectures that $\{g_k\}_{k \in \mathbb{Z}}$ is a tight frame for $L^2(B)$ with frame bound $T$. Let $T : \ell^2(B) \to \ell^2(\mathbb{Z})$, and $\hat{T} : \ell^2(B) \to \ell^2(\mathbb{Z})$ be the analysis operators corresponding respectively to the frame $\{g_k\}_{k \in \mathbb{Z}}$ and to the canonical dual frame $\{\hat{g}_k\}_{k \in \mathbb{Z}}$. Next, let $\text{arb} : \mathbb{R} \to \mathbb{C}$ be an arbitrary bounded function, and define the functions $h_{\text{LP}}, h_{\text{out}} : \mathbb{R} \to \mathbb{C}$ by specifying their $L^2$-Fourier transforms according to

$$h_{\text{LP}}(f) = \begin{cases} 1, & |f| \leq BT \\ 0, & \text{else} \end{cases}, \quad \text{and} \quad h_{\text{out}}(f) = \begin{cases} \text{arb}(f), & BT < |f| \leq \frac{1}{2} \\ 0, & \text{else} \end{cases}.$$  

Consider the operators $A : \ell^2(\mathbb{Z}) \to \ell^2(B)$ and $B : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$ defined as

$$A : \{c_k\}_{k \in \mathbb{Z}} \mapsto \sum_{k \in \mathbb{Z}} c_k h_{\text{LP}}(\frac{t}{T} - k), \quad \text{and} \quad B : \{c_k\}_{k \in \mathbb{Z}} \mapsto \sum_{k \in \mathbb{Z}} c_k h_{\text{out}}(\frac{t}{T} - k).$$  

(i) For a sequence $\{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)$, show that $A \{a_k\}_{k \in \mathbb{Z}} = \hat{T}^* \{a_k\}_{k \in \mathbb{Z}}$.  

(ii) For a sequence $\{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)^\perp$, show that $A \{b_k\}_{k \in \mathbb{Z}} = 0$.  

(iii) Using the fact that every sequence $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ can be decomposed as $\{c_k\}_{k \in \mathbb{Z}} = \{a_k\}_{k \in \mathbb{Z}} + \{b_k\}_{k \in \mathbb{Z}}$ with $\{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)$ and $\{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)^\perp$, show that $A = \hat{T}^* B$, where $\mathcal{P} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the orthogonal projection onto $\mathcal{R}(T)$.  

(iv) Use the ideas in items (i) — (iii) to show that $B = B(\text{Id}_{\ell^2(\mathbb{Z})} - \mathcal{P})$.

Problem 3  Frames for $\mathbb{C}^M$
Assume that $\{f_k\}_{k=1}^N$ is a frame for $\mathbb{C}^M$. Prove that the $2N$ vectors consisting of the real parts and of the imaginary parts of the frame vectors constitute a frame for $\mathbb{R}^M$.  

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Homework 2
Problem 4  Tight frames
Let \( \{ f_k \}_{k=0}^{\infty} \) be a frame for the Hilbert space \( \mathcal{H} \). Show that the following statements are equivalent:

(i) \( \{ f_k \}_{k=0}^{\infty} \) is tight

(ii) \( \{ f_k \}_{k=0}^{\infty} \) has a dual of the form \( g_k = C f_k \) for some constant \( C > 0 \).

Problem 5  Unitary transformation of a frame
Let \( \{ f_k \}_{k \in \mathcal{K}} \) be a frame for a Hilbert space \( \mathcal{H} \) with frame bounds \( A \) and \( B \). Let \( U : \mathcal{H} \rightarrow \mathcal{H} \) be a unitary operator. Show that the set \( \{ U f_k \}_{k \in \mathcal{K}} \) is again a frame for \( \mathcal{H} \) and compute the corresponding frame bounds.

Problem 6  Redundancy of a frame
Let \( \{ f_k \}_{k=1}^{N} \) be a frame for \( \mathbb{C}^M \) with \( N > M \). Assume that the frame elements are normalized such that \( \| f_k \| = 1 \) for all \( k \). The ratio \( N/M \) is called redundancy of the frame.

a) Assume that \( \{ f_k \}_{k=1}^{N} \) is a tight frame with frame bound \( A \). Show that \( A = N/M \).

b) Now assume that \( A \) and \( B \) are lower and upper frame bounds of \( \{ f_k \}_{k=1}^{N} \), respectively. Show that \( A \leq N/M \leq B \).

Problem 7  Frame expansion with noise
Let \( \{ g_j \}_{j=1}^{M} \) be a tight frame for \( \mathbb{C}^N \) (\( N \leq M \)) with \( \| g_j \| = 1 \) for all \( 1 \leq j \leq M \). We know that every \( f \in \mathbb{C}^N \) can be perfectly reconstructed from its frame expansion coefficients according to

\[
 f = \frac{1}{A} \sum_{j=1}^{M} \langle f, g_j \rangle g_j, 
\]

where \( A = M/N \). Now assume that the frame expansion coefficients are subject to noise:

\[
 \langle f, g_j \rangle \rightarrow \langle f, g_j \rangle + w_j 
\]

where \( \{ w_j \}_{j=1}^{M} \) are independent, zero-mean random variables with variance \( N_0 \) each. After reconstruction we obtain in this case

\[
 f_w = \frac{1}{A} \sum_{j=1}^{M} (\langle f, g_j \rangle + w_j) g_j.
\]

Compute the mean squared error (MSE) of the noisy reconstruction, defined as \( \mathbb{E}\{ \| f - f_w \|^2 \} \). How does the MSE depend on the redundancy \( r = M/N \)? Give an interpretation of the result.