Problem 1  Change of basis matrix between ONBs is unitary

We have to verify that $U^* U = U U^* = \text{Id}$. First note that, for $j, k \in [N]$, we have

$$
\langle h_k, h_j \rangle = \left( \sum_{\ell=1}^{N} \langle h_k, g_{\ell} \rangle g_{\ell} \right) \left( \sum_{\ell' = 1}^{N} \langle h_j, g_{\ell'} \rangle g_{\ell'} \right) = \sum_{\ell=1}^{N} \sum_{\ell' = 1}^{N} \langle h_k, g_{\ell} \rangle \langle h_j, g_{\ell'} \rangle \langle g_{\ell}, g_{\ell'} \rangle = \delta_{\ell \ell'},
$$

Therefore, for $j, k \in [N]$, we have

$$
[U^* U]_{jk} = \sum_{\ell=1}^{N} [U^*]_{j \ell} U_{\ell k} = \sum_{\ell=1}^{N} U_{\ell j} U_{\ell k} = \sum_{\ell=1}^{N} \langle h_j, g_{\ell} \rangle \langle h_k, g_{\ell} \rangle = \delta_{jk},
$$

and so $U^* U = \text{Id}$. A completely analogous computation shows that $U U^* = \text{Id}$.

Problem 2  Oversampled A/D conversion

(i) Take $\{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(\mathbb{T})$. By definition of $A$, we have

$$
A \{a_k\}_{k \in \mathbb{Z}} = \sum_{k \in \mathbb{Z}} a_k h_{\text{LP}} \left( \frac{t}{T} - k \right).
$$

The result now follows immediately by noting that

$$
\hat{T}^* \{a_k\}_{k \in \mathbb{Z}} = \sum_{k \in \mathbb{Z}} a_k \hat{g}_k(t),
$$

where $\hat{g}_k(t) = T g_k(t) = h_{\text{LP}} \left( \frac{t}{T} - k \right)$. 

(ii) Take \( \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)^\perp \). Then,
\[
A\{b_k\}_{k \in \mathbb{Z}} = \sum_{k \in \mathbb{Z}} b_k h_{\text{LP}} \left( \frac{t}{T} - k \right) \\
= \sum_{k \in \mathbb{Z}} b_k \int_{-1/2}^{1/2} \hat{h}_{\text{LP}}(f) e^{2\pi i f(t/T - k)} df \\
= \int_{-1/2}^{1/2} \hat{h}_{\text{LP}}(f) e^{2\pi i f/T} \sum_{k = -\infty}^{\infty} b_k e^{-2\pi ikf} df \\
= \int_{-1/2}^{1/2} \hat{b}(f) \hat{h}_{\text{LP}}(f) e^{2\pi i f/T} df \\
= 0, \tag{1}
\]
where the last equality follows because \( \hat{b}(f) \) is supported on the set \([-1/2, -BT] \cup [BT, 1/2] \) and \( \hat{h}_{\text{LP}}(f) \) is supported on the set \([-BT, BT] \).

(iii) Take \( \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \). We can write \( \{c_k\}_{k \in \mathbb{Z}} = \{a_k\}_{k \in \mathbb{Z}} + \{b_k\}_{k \in \mathbb{Z}} \) with \( \{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T) \) and \( \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)^\perp \). It was proved above that \( A\{a_k\}_{k \in \mathbb{Z}} = \hat{T}^*\{a_k\}_{k \in \mathbb{Z}} \) and \( A\{b_k\}_{k \in \mathbb{Z}} = 0 \). Therefore,
\[
A\{c_k\}_{k \in \mathbb{Z}} = A\{a_k\}_{k \in \mathbb{Z}} + \underbrace{A\{b_k\}_{k \in \mathbb{Z}}}_{0} = \hat{T}^*\{a_k\}_{k \in \mathbb{Z}}.
\]
Since \( \{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T) \) and \( \mathbb{P} \) is the orthogonal projection operator onto \( \mathcal{R}(T) \), we have \( \mathbb{P}\{a_k\}_{k \in \mathbb{Z}} = \{a_k\}_{k \in \mathbb{Z}} \). Similarly, since \( \{b_k\}_{k \in \mathbb{Z}} \) is in the orthogonal complement of \( \mathcal{R}(T) \), we have \( \mathbb{P}\{b_k\}_{k \in \mathbb{Z}} = 0 \). Therefore,
\[
\hat{T}^*\mathbb{P}\{c_k\}_{k \in \mathbb{Z}} = \hat{T}^*\underbrace{\mathbb{P}\{a_k\}_{k \in \mathbb{Z}}}_{\{a_k\}_{k \in \mathbb{Z}}} + \hat{T}^*\underbrace{\mathbb{P}\{b_k\}_{k \in \mathbb{Z}}}_{0} = \hat{T}^*\{a_k\}_{k \in \mathbb{Z}}.
\]
We conclude that \( A = \hat{T}^*\mathbb{P} \), as required.

(iv) Take \( \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T) \). Then,
\[
B\{b_k\}_{k \in \mathbb{Z}} = \sum_{k = -\infty}^{\infty} b_k h_{\text{out}} \left( \frac{t}{T} - k \right) \\
= \int_{-1/2}^{1/2} \hat{b}(f) \hat{h}_{\text{out}}(f) e^{2\pi i f/T} df \\
= 0,
\]
where the third equality follows because \( \hat{b}(f) \) is supported on the set \([-BT, BT] \) and \( \hat{h}_{\text{out}}(f) \) is supported on the set \([-1/2, -BT] \cup [BT, 1/2] \).

Next, take \( \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \). We can write \( \{c_k\}_{k \in \mathbb{Z}} = \{a_k\}_{k \in \mathbb{Z}} + \{b_k\}_{k \in \mathbb{Z}} \) with \( \{a_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T)^\perp \) and \( \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T) \). Then, as \( \mathbb{B}\{b_k\}_{k \in \mathbb{Z}} = 0 \), we have that
\[
\mathbb{B}\{c_k\}_{k \in \mathbb{Z}} = \mathbb{B}\{a_k\}_{k \in \mathbb{Z}} + \underbrace{\mathbb{B}\{b_k\}_{k \in \mathbb{Z}}}_{0} = \mathbb{B}\{a_k\}_{k \in \mathbb{Z}}. \tag{1}
\]
Since \( \{a_k\}_{k \in \mathbb{Z}} \) is in the orthogonal complement of \( \mathcal{R}(T) \) and \( \mathbb{P} \) is the orthogonal projection operator onto \( \mathcal{R}(T) \), we have \( \mathbb{P}\{a_k\}_{k \in \mathbb{Z}} = 0 \), or, equivalently, \( (\text{Id}_{\ell^2(\mathbb{Z})} - \mathbb{P})\{a_k\}_{k \in \mathbb{Z}} = \{a_k\}_{k \in \mathbb{Z}} \). Similarly, since \( \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{R}(T) \), we have \( \mathbb{P}\{b_k\}_{k \in \mathbb{Z}} = \{b_k\}_{k \in \mathbb{Z}} \), or, equivalently \( (\text{Id}_{\ell^2(\mathbb{Z})} - \mathbb{P})\{b_k\}_{k \in \mathbb{Z}} = 0 \).
Therefore,
\[
\mathbb{E}(\text{Id}_{L^2(\mathbb{Z})} - P)\{c_k\}_{k \in \mathbb{Z}} = \mathbb{E}(\text{Id}_{L^2(\mathbb{Z})} - P)\{a_k\}_{k \in \mathbb{Z}} + \mathbb{E}(\text{Id}_{L^2(\mathbb{Z})} - P)\{b_k\}_{k \in \mathbb{Z}} = \mathbb{E}\{a_k\}_{k \in \mathbb{Z}}.
\] (2)

Comparing (1) and (2), we can therefore conclude that \( \mathbb{E} = \mathbb{E}(\text{Id}_{L^2(\mathbb{Z})} - P) \), as desired.

**Problem 3  Frames for \(\mathbb{C}^M\)**

Assume that \(\{f_k\}_{k=1}^N\) is a frame for \(\mathbb{C}^M\) with frame bounds \(A, B > 0\) and set
\[
g_k = \begin{cases} 
\text{Re}\{f_k\}, & 1 \leq k \leq N \\
\text{Im}\{f_{k-N}\}, & N + 1 \leq k \leq 2N.
\end{cases}
\]

For all \(f \in \mathbb{R}^M\), we then have
\[
\sum_{k=1}^{2N} |\langle f, g_k \rangle|^2 = \sum_{k=1}^{N} |\langle f, \text{Re}\{f_k\} \rangle|^2 + |\langle f, \text{Im}\{f_k\} \rangle|^2 = \sum_{k=1}^{N} |\langle f, \text{Re}\{f_k\} \rangle - i\langle f, \text{Im}\{f_k\} \rangle|^2
\]
\[
= \sum_{k=1}^{N} |\langle f, \text{Re}\{f_k\} + i\text{Im}\{f_k\} \rangle|^2 = \sum_{k=1}^{N} |\langle f, f_k \rangle|^2,
\]
which implies that \(\{g_k\}_{k=1}^{2N}\) is a frame for \(\mathbb{R}^M\) with frame bounds \(A, B > 0\).

**Problem 4  Tight frames**

Let \(\{f_k\}_{k=0}^\infty\) be a frame for the Hilbert space \(\mathcal{H}\).

- Assume that \(\{f_k\}_{k=0}^\infty\) is tight. Then there exists a constant \(A > 0\) such that
\[
\sum_{k=0}^\infty |\langle f, f_k \rangle|^2 = A\|f\|^2
\]
for all \(f \in \mathcal{H}\). We can define \(g_k = A^{-1}f_k\) for all \(k \in \mathbb{N}\). We have then
\[
\sum_{k=0}^\infty \langle f, g_k \rangle f_k = \sum_{k=0}^\infty \langle f, f_k \rangle g_k = A^{-1} \sum_{k=0}^\infty \langle f, f_k \rangle f_k = f,
\]
where we used the fact that the frame operator \(S\) satisfies \(S = A\|f\|^2\) since \(\{f_k\}_{k=0}^\infty\) is a tight frame with frame bound \(A\). Therefore, \(\{g_k\}_{k=0}^\infty\) forms a dual frame\(^1\) of \(\{f_k\}_{k=0}^\infty\).

- Conversely, assume that \(\{f_k\}_{k=0}^\infty\) has a dual of the form \(g_k = Cf_k\) with \(C > 0\). Then for all \(f \in \mathcal{H}\), we have
\[
f = \sum_{k=0}^\infty \langle f, g_k \rangle f_k = \sum_{k=0}^\infty \langle f, f_k \rangle g_k = C \sum_{k=0}^\infty \langle f, f_k \rangle f_k = CSf,
\]
which shows that the frame operator is \(S = C^{-1}I\), and that \(\{f_k\}_{k=0}^\infty\) is hence a tight frame.

\(^1\)Note that \(\{g_k\}_{k=0}^\infty\) is in fact the canonical dual frame of \(\{f_k\}_{k=0}^\infty\), since \(g_k = S^{-1}f_k\) for all \(k \in \mathbb{N}\).
Problem 5  Unitary transformation of a frame

Since \( \{f_k\}_{k \in K} \) is a frame with frame bounds \( A \) and \( B \), we have
\[
A \|f\|^2 \leq \sum_{k \in K} |\langle f, f_k \rangle|^2 \leq B \|f\|^2.
\]

Moreover, since \( U \) is a unitary operator, one has \( U^* U = UU^* = I \). Hence, we have
\[
\|U^* f\|^2 = \langle U^* f, U^* f \rangle = \langle UU^* f, f \rangle = \langle f, f \rangle = \|f\|^2.
\]

We have then
\[
\sum_{k \in K} |\langle f, U f_k \rangle|^2 = \sum_{k \in K} |\langle U^* f, f_k \rangle|^2 \leq B \|U^* f\|^2 = B \|f\|^2,
\]
which establishes the upper frame bound. Next,
\[
\sum_{k \in K} |\langle f, U f_k \rangle|^2 = \sum_{j \in K} |\langle U^* f, f_k \rangle|^2 \geq A \|U^* f\|^2 = A \|f\|^2,
\]
which establishes the lower frame bound. Therefore, \( \{U f_k\}_{k \in K} \) is a frame for \( \mathcal{H} \) with the same frame bounds as \( \{f_k\}_{k \in K} \).

Problem 6  Redundancy of a frame

a) Assume that \( \{f_k\}_{k=1}^N \) is a tight frame for \( \mathbb{C}^M \) with frame bound \( A \) such that \( \|f_k\| = 1 \) for all \( 1 \leq k \leq N \). Choose an orthonormal basis \( \{e_k\}_{k=1}^M \) for \( \mathbb{C}^M \). Using Parseval’s equality and the fact that the \( f_k, 1 \leq k \leq N \), are normalized, we obtain
\[
1 = \|f_k\|^2 = \sum_{\ell=1}^M |\langle f_k, e_\ell \rangle|^2
\]
for all \( 1 \leq k \leq N \). On the other hand, since \( \{f_k\}_{k=1}^N \) is a tight frame with frame bound \( A \), we have
\[
A = A \|e_\ell\|^2 = \sum_{k=1}^N |\langle e_\ell, f_k \rangle|^2
\]
(1)
for all \( 1 \leq \ell \leq M \). Summing (1) for all \( 1 \leq \ell \leq M \) yields
\[
MA = \sum_{\ell=1}^M \sum_{k=1}^N |\langle e_\ell, f_k \rangle|^2 = \sum_{k=1}^N \sum_{\ell=1}^M |\langle f_k, e_\ell \rangle|^2 = N.
\]

As a result, we have necessarily \( A = N/M \).

b) Assume that \( \{f_k\}_{k=1}^N \) is a frame for \( \mathbb{C}^M \) with frame bounds \( A \) and \( B \) such that \( \|f_k\| = 1 \) for all \( 1 \leq k \leq N \). As in 1., choose an orthonormal basis \( \{e_k\}_{k=1}^M \) for \( \mathbb{C}^M \). Again, using Parseval’s equality and the fact that the \( f_k, 1 \leq k \leq N \), are normalized, we obtain
\[
1 = \|f_k\|^2 = \sum_{\ell=1}^M |\langle f_k, e_\ell \rangle|^2
\]
for all \( 1 \leq k \leq N \). Since \( \{f_k\}_{k=1}^N \) is a frame for \( \mathbb{C}^M \) with frame bounds \( A, B \), we have for
all \( 1 \leq \ell \leq M \) that
\[
A = A\|e_\ell\|^2 \leq \sum_{k=1}^{N} |\langle e_\ell, f_k \rangle|^2 \leq B\|e_\ell\|^2 = B. \tag{2}
\]

Summing (2) for all \( 1 \leq \ell \leq M \) then gives
\[
AM \leq \sum_{\ell=1}^{M} \sum_{n=1}^{N} |\langle e_\ell, f_k \rangle|^2 = \sum_{n=1}^{N} \sum_{\ell=1}^{M} |\langle f_k, e_\ell \rangle|^2 \leq BM,
\]

which results in \( A \leq N/M \leq B \).

**Problem 7  Frame expansion with noise**

We have the following:
\[
E\|f - f_w\|^2 = E\left\| \frac{1}{A} \sum_{j=1}^{M} w_j g_j \right\|^2 = E\frac{1}{A^2} \sum_{j=1}^{M} \sum_{\ell=1}^{M} w_j w_\ell \langle g_j \rangle \langle g_\ell \rangle = N_0 \frac{1}{A^2} \sum_{j=1}^{M} \|g_j\|^2 = N_0 M = N_0 N \frac{1}{r}.
\]

For any Hilbert space of dimension \( N \), the MSE is inversely proportional to the redundancy. Therefore, it is an advantage to formulate algorithms involving frames than bases, which have redundancy \( r = 1 \).