Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Dr. Helmut Bölcskei
Chair for Mathematical Information Science

Solutions to the **Exam on Neural Network Theory** **February 11, 2021**

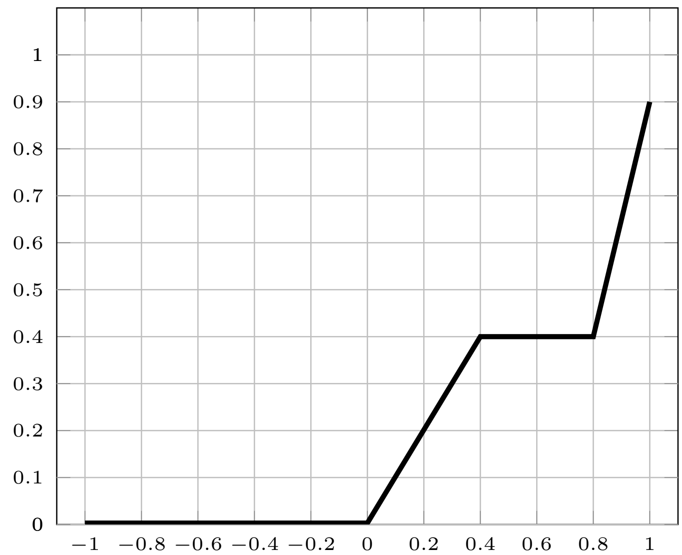
Problem 1

(a) Note that $|x| = \rho(x) + \rho(-x)$ and $10x = 1.25\rho \circ 2\rho \circ 2\rho \circ 2\rho(x)$, $\forall x \geq 0$. Since $|x| \geq 0$, the function $f(x) = 10|x|$ can hence be realized through the network

$$\Psi(x) = 1.25 \circ \rho \circ 2 \circ \rho \circ 2 \circ \rho \circ \begin{pmatrix} 2 & 2 \end{pmatrix} \circ \rho \circ \begin{pmatrix} 1 \\ -1 \end{pmatrix} x.$$

This network satisfies $\mathcal{W}(\Psi) = 2$, $\mathcal{B}(\Psi) = 2$, and has depth $\mathcal{L}(\Psi) = 5$ and connectivity $\mathcal{M}(\Psi) = 7$.

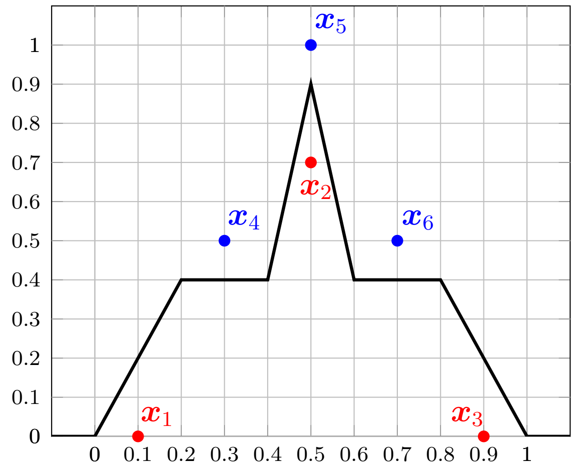
(b) The network Φ consists of two affine maps, its depth is therefore 2, the width is 3 and the connectivity is 8. The plot of $\Phi(x)$ on the interval $[-1, 1]$ is depicted below.



(c) The function $f(x)$ can be inferred from its definition according to $f(x) = \Phi(2x)$ on $[-\infty, 1/2]$ and by exploiting symmetry. In summary, we get

$$f(x) = \begin{cases} \Phi(2x), & \text{if } x \leq 1/2, \\ \Phi(2 - 2x), & \text{if } x > 1/2, \end{cases}$$

where $\Phi(x)$ is the ReLU network from subproblem (b). The following figure depicting $f(x)$ shows that $f(x)$ is class-separating for the problem at hand.



Denote the ReLU function as $\rho(x) = \max\{0, x\}$. The function $f(x)$ can be realized according to

$$\begin{aligned} f(x) = \rho(2x) - \rho(2x - 0.4) + \rho(5x - 2) - \rho(10x - 5) \\ + \rho(5x - 3) - \rho(2x - 1.6) + \rho(2x - 2). \end{aligned}$$

The corresponding depth-2 ReLU network is given by

$$\Phi(x) = \big(W_2 \circ \rho \circ W_1\big)(x)$$

with

$$W_1(x) = \begin{pmatrix} 2 \\ 2 \\ 5 \\ 10 \\ 5 \\ 2 \\ 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -0.4 \\ -2 \\ -5 \\ -3 \\ -1.6 \\ -2 \end{pmatrix}, \quad W_2(x) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}.$$

The network Φ has depth 2, width 7, and connectivity 20.

(d) The sawtooth functions $g_s(x)$ are periodic with period 2^{-s+1}, see Figure 1. Thus, the cardinality of the set $\{x : g_s(x) = 1,\ x \in [0, 1]\}$ is 2^{s-1}.

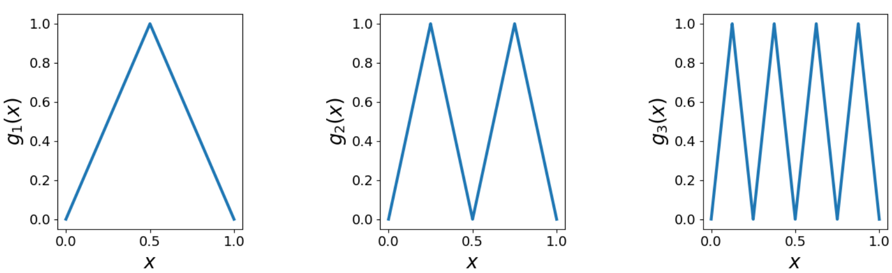


Figure 1: Sawtooth functions

(e) Note that

$$g(x) = \rho(2x) - \rho(4x-2) + \rho(2x-2) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/2, \\ 2-2x, & \text{if } 1/2 < x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

It therefore follows that

$$f(x) = \big(\Phi \circ g\big)(x),$$

where we also used that $\Phi(x) = 0, \forall x \le 0$. The sawtooth function $g(x)$ can be realized through a ReLU network according to $g(x) = \rho(2x) - \rho(4x-2) + \rho(2x-2) = \big(W_2^g \circ \rho \circ W_1^g\big)(x)$ with

$$W_1^g(x) = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}, \quad W_2^g(x) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Now, recall that the network $\Phi(x)$ is given by $\Phi(x) = \big(W_2 \circ \rho \circ W_1\big)(x)$ with

$$W_1(x) = \begin{pmatrix} 1 \\ 1 \\ 2.5 \end{pmatrix} x - \begin{pmatrix} 0 \\ 0.4 \\ 2 \end{pmatrix}, \quad W_2(x) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The composition $f(x) = \big(\Phi \circ g\big)(x)$ can hence be realized through the network

$$\Phi_2(x) = \big(W_2 \circ \rho \circ W_1 W_2^g \circ \rho \circ W_1^g\big)(x) = \big(W_3' \circ \rho \circ W_2' \circ \rho \circ W_1'\big)(x),$$

where

$$W_1'(x) = W_1^g(x) = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}, \quad W_3'(x) = W_2(x) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{aligned} W_2'(x) = \big(W_1 W_2^g\big)(x) &= \begin{pmatrix} 1 \\ 1 \\ 2.5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 2.5 & -2.5 & 2.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.4 \\ 2 \end{pmatrix}. \end{aligned}$$

The network Φ_2 has depth 3, width 3, and connectivity 19.

Problem 2

(a) (i) An ϵ-covering of the compact set $\mathcal{C} \subseteq \mathcal{X}$ with respect to the metric ρ is a set $\{x_1, \ldots, x_N\} \subset \mathcal{C}$ such that for each $x \in \mathcal{C}$, there exists an $i \in [1, N]$ so that $\rho(x, x_i) \leq \epsilon$. The ϵ-covering number $N(\epsilon; \mathcal{C}, \rho)$ is the cardinality of a smallest ϵ-covering of $\mathcal{C}$.

(ii) An ϵ-packing of a compact set $\mathcal{C} \subseteq \mathcal{X}$ with respect to the metric ρ is a set $\{x_1, \ldots, x_N\} \subset \mathcal{C}$ such that $\rho(x_i, x_j) > \epsilon$, for all distinct i, j. The ϵ-packing number $M(\epsilon; \mathcal{C}, \rho)$ is the cardinality of a largest ϵ-packing of $\mathcal{C}$.

(iii) $N(2\epsilon; \mathcal{C}, \rho) \leq M(2\epsilon; \mathcal{C}, \rho) \leq N(\epsilon; \mathcal{C}, \rho) \leq M(\epsilon; \mathcal{C}, \rho)$.

(b) (i) For every $f_\theta \in \mathcal{F}$, we can find a θ_i in the set $\{\theta_0, \ldots, \theta_T, \theta_{T+1}\}$, such that $|\theta_i - \theta| \leq \epsilon$. We then have

$$\begin{aligned}\|f_{\theta_i} - f_\theta\|_\infty &= \max_{x\in[0,1]} \Big| \ln(1+\theta_i x) - \ln(1+\theta x) \Big| = \max_{x\in[0,1]} \left| \ln\left(\frac{1+\theta_i x}{1+\theta x}\right)\right| \\ &= \max_{x\in[0,1]} \left| \ln\left(1 + \frac{(\theta_i - \theta)x}{1+\theta x}\right)\right| \leq \max_{x\in[0,1]} \left|\frac{(\theta_i-\theta)x}{1+\theta x}\right| \leq |\theta_i - \theta| \leq \epsilon.\end{aligned}$$

Therefore, we can conclude that the set $\{f_{\theta_0}, \ldots, f_{\theta_T}, f_{\theta_{T+1}}\}$ constitutes an ϵ-covering of $\mathcal{F}$. An upper bound on the covering number is hence given by $N(\epsilon; \mathcal{F}, \|\cdot\|_\infty) \leq T + 2 \leq \frac{1}{2\epsilon} + 2$.

(ii) We construct an explicit packing as follows. Set $T = \lfloor \frac{1}{3\epsilon} \rfloor$, and for $i = 0, 1, \ldots, T$, define $\theta_i = 3\epsilon i$. Moreover, note that for all i, j with $i \neq j$, we have

$$\begin{aligned}\|f_{\theta_i} - f_{\theta_j}\|_\infty &= \max_{x\in[0,1]} \Big| \ln(1+\theta_i x) - \ln(1+\theta_j x) \Big| = \max_{x\in[0,1]} \left| \ln\left(\frac{1+\theta_i x}{1+\theta_j x}\right)\right| \\ &= \max_{x\in[0,1]} \left| \ln\left(1 + \frac{(\theta_i - \theta_j)x}{1+\theta_j x}\right)\right| \geq \max_{x\in[0,1]} \left| \left(\frac{(\theta_i-\theta_j)x}{1+\theta_j x}\right) \Big/ \left(1 + \frac{(\theta_i-\theta_j)x}{1+\theta_j x}\right)\right| \\ &= \max_{x\in[0,1]} \left| \left(\frac{(\theta_i-\theta_j)x}{1+\theta_j x}\right) \Big/ \left(\frac{1+\theta_j x + \theta_i x - \theta_j x}{1+\theta_j x}\right)\right| = \max_{x\in[0,1]} \left|\frac{(\theta_i-\theta_j)x}{1+\theta_i x}\right| \\ &\geq \max_{x\in[0,1]} \left|\frac{(\theta_i-\theta_j)x}{2}\right| = \left|\frac{\theta_i-\theta_j}{2}\right| = \left|\frac{3\epsilon(i-j)}{2}\right| > \epsilon,\end{aligned}$$

by definition of θ_i. We can therefore conclude that $\{f_{\theta_0}, \ldots, f_{\theta_T}\}$ is an ϵ-packing and the corresponding packing number satisfies $M(\epsilon; \mathcal{F}, \|\cdot\|_\infty) \geq T + 1 \geq \frac{1}{3\epsilon}$.

(iii) By subproblems (a.iii), (b.i), and (b.ii), we obtain

$$\frac{1}{6\epsilon} \leq M(2\epsilon; \mathcal{F}, \|\cdot\|_\infty) \leq N(\epsilon; \mathcal{F}, \|\cdot\|_\infty) \leq \frac{1}{2\epsilon} + 2,$$

which allows us to conclude that $\log N(\epsilon; \mathcal{F}, \|\cdot\|_\infty) \asymp \log(1/\epsilon)$, as $\epsilon \to 0$.

Problem 3

(a) As a and $T_b(a)$ differ only in digits after the b-th position in the fractional parts of their binary representation, we have

$$|a - \tilde{a}| \le \sum_{i=b+1}^{\infty} 2^{-i} = 2^{-b} \sum_{i=1}^{\infty} 2^{-i} = 2^{-b},$$

where we used $\sum_{i=1}^{\infty} 2^{-i} = 1$ in the last equality.

(b) As T_b acts entrywise on A_1, we have

$$\left\| A_1 - \tilde{A}_1 \right\|_\infty = \|A_1 - T_b(A_1)\|_\infty \le \sup_{a \in \mathbb{R}} |a - T_b(a)| \le 2^{-b}. \tag{1}$$

We therefore get

$$\sup_{x \in [-1,1]^d} \left\| Ax - \tilde{A}_1 x \right\|_\infty = \sup_{x \in [-1,1]^d} \left\| \left(A_1 - \tilde{A}_1 \right) x \right\|_\infty \tag{2}$$

$$\le \sup_{x \in [-1,1]^d} d \left\| A_1 - \tilde{A}_1 \right\|_\infty \|x\|_\infty \tag{3}$$

$$\le d\, 2^{-b}, \tag{4}$$

where in (3) we used (1) and the inequality in the hint.

(c) The first derivative of σ is given by $\sigma'(x) = \frac{e^x}{(1+e^x)^2}$, $x \in \mathbb{R}$, which is positive for all $x \in \mathbb{R}$. It therefore follows that σ is strictly increasing, and hence $\sup_{x \in \mathbb{R}} \sigma(x) \le \lim_{x \to \infty} \sigma(x) = 1$ and $\inf_{x \in \mathbb{R}} \sigma(x) \ge \lim_{x \to -\infty} \sigma(x) = 0$. The second derivative of σ is given by $\sigma''(x) = \frac{e^x(1-e^x)}{(1+e^x)^3}$, $x \in \mathbb{R}$, which is positive on $(-\infty, 0)$ and negative on $(0, \infty)$. This implies that σ' is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Hence σ' attains its maximum at $x = 0$ with $\sigma'(0) = \frac{1}{4}$. For every $x, y \in \mathbb{R}$, by the mean value theorem, there exists a real number z between x and y such that

$$\sigma(x) - \sigma(y) = \sigma'(z)(x - y),$$

which implies

$$|\sigma(x) - \sigma(y)| = |\sigma'(z)(x - y)| \le \frac{1}{4} |x - y|,$$

where we used $0 < \sigma'(z) \le \sigma'(0) = \frac{1}{4}$.

(d) We have

$$\sup_{x \in [-1,1]^d} \left\| \sigma\big(A_1 x\big) - \sigma\big(\tilde{A}_1 x\big) \right\|_\infty \tag{5}$$

$$\le \sup_{x \in [-1,1]^d} \frac{1}{4} \left\| A_1 x - \tilde{A}_1 x \right\|_\infty \tag{6}$$

$$\le \frac{1}{4} d\, 2^{-b}, \tag{7}$$

where in (6) we used the definition of the $\|\cdot\|_\infty$-norm and the Lipschitz continuity established in subproblem (c), and (7) follows from subproblem (b).

(e) We have, for all $x \in [-1,1]^d$,

$$
\begin{aligned}
&\left|\Phi(x) - \tilde{\Phi}(x)\right| && (8)\\
&= \left\|A_2\sigma(A_1x) - \tilde{A}_2\sigma(\tilde{A}_1x)\right\|_\infty && (9)\\
&= \left\|A_2\sigma\big(A_1x\big) - A_2\sigma\big(\tilde{A}_1x\big) + A_2\sigma\big(\tilde{A}_1x\big) - \tilde{A}_2\sigma\big(\tilde{A}_1x\big)\right\|_\infty && (10)\\
&\le \left\|A_2\left(\sigma\big(A_1x\big) - \sigma\big(\tilde{A}_1x\big)\right)\right\|_\infty + \left\|\big(A_2 - \tilde{A}_2\big)\sigma\big(\tilde{A}_1x\big)\right\|_\infty && (11)\\
&\le N\left\|A_2\right\|_\infty \left\|\sigma\big(A_1x\big) - \sigma\big(\tilde{A}_1x\big)\right\|_\infty + N\left\|A_2 - \tilde{A}_2\right\|_\infty \left\|\sigma\big(\tilde{A}_1x\big)\right\|_\infty && (12)\\
&\le \frac{1}{4}Nd\,2^{-b} + N2^{-b}, && (13)
\end{aligned}
$$

where in (12) we used the inequality derived in (3), and in (13) we employed $\|A_2\|_\infty \le 1$, the result from subproblem (d), and $\left\|\sigma\big(\tilde{A}_1x\big)\right\|_\infty \le 1$.

Problem 4

(a) The dichotomy $\{X_1^+, X_1^-\}$ is said to be homogeneously linearly separable if there exists a nonzero vector $w_1 \in \mathbb{R}^d$ such that

$$\langle x, w_1 \rangle > 0, \text{ for all } x \in X_1^+,$$
$$\langle x, w_1 \rangle < 0, \text{ for all } x \in X_1^-,$$

and it is said to be ϕ-separable if there exists a nonzero vector $w_2 \in \mathbb{R}^m$ such that

$$\langle \phi(x), w_2 \rangle > 0, \text{ for all } x \in X_1^+,$$
$$\langle \phi(x), w_2 \rangle < 0, \text{ for all } x \in X_1^-.$$

Since there are at most $2^{\text{card}(X_1)}$ dichotomies of X_1, as explained in the lecture, it follows from the inclusion relation that the number of homogeneously linearly separable dichotomies of X_1 is less than or equal to $2^{\text{card}(X_1)}$, with equality if every dichotomy of X_1 is homogeneously linearly separable.

(b) Let S_{X_1}, $S_{X_1 \cup \{x_1\}}$ be the sets of homogeneously linearly separable dichotomies of X_1 and $X_1 \cup \{x_1\}$, respectively. Consider a dichotomy $\{X^+, X^-\} \in S_{X_1 \cup \{x_1\}}$ and note that $\{X^+, X^-\}$ can be written as

$$\left\{X_1^+ \cup \{x_1\}, X_1^-\right\} \text{ or } \left\{X_1^+, X_1^- \cup \{x_1\}\right\} \tag{14}$$

for some dichotomy $\{X_1^+, X_1^-\}$ of X_1. As $X_1^+ \subset X^+$ and $X_1^- \subset X^-$, the dichotomy $\{X_1^+, X_1^-\}$ can be separated by the hyperplane that separates $\{X^+, X^-\}$. Therefore, the dichotomy $\{X_1^+, X_1^-\}$ in (14) is homogeneously linearly separable and we have

$$S_{X_1 \cup \{x_1\}} \subset \left\{\left\{X_1^+ \cup \{x_1\}, X_1^-\right\} \text{ or } \left\{X_1^+, X_1^- \cup \{x_1\}\right\} : \left\{X_1^+, X_1^-\right\} \in S_{X_1}\right\},$$

which implies

$$\text{card}\big(S_{X_1 \cup \{x_1\}}\big) \leq 2 \,\text{card}\big(S_{X_1}\big) = 2C.$$

(c) The dichotomy is not homogeneously linearly separable as there is no line through the origin such that $(-1, 0)$ and $(1, 0)$ lie on the same side of the line. Otherwise, there would exist a nonzero vector $w = \big(w_1, w_2\big) \in \mathbb{R}^2$ such that $\langle w, (1, 0) \rangle = w_1 > 0$ and $\langle w, (-1, 0) \rangle = -w_1 < 0$, which constitutes a contradiction. See Fig. 2 for an illustration.

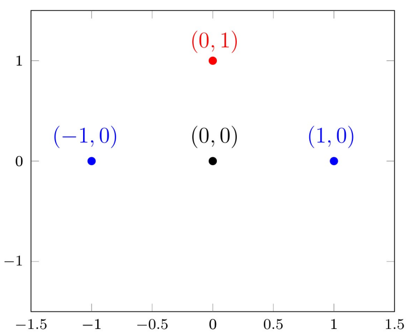


Figure 2: If the dichotomy were homogeneously linearly separable, there would exist a line through the origin that separates the blue points from the red point.

(d) The dichotomy is ϕ_1-separable. Let $w = (1, -1)$. Then, $\langle \phi_1(1,0), w \rangle = 1 > 0$, $\langle \phi_1(-1,0), w \rangle = 1 > 0$, and $\langle \phi_1(0,1), w \rangle = -1 < 0$.

(e) Suppose for the sake of contradiction that the dichotomy $\left\{X_3^+, X_3^-\right\}$ is ϕ_2-separable. Then, there would exist a nonzero vector $w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ such that

$$\begin{aligned} \langle \phi_2(x,y), w \rangle > 0, &\quad \text{for all } (x,y) \in X_3^+, \\ \langle \phi_2(x,y), w \rangle < 0, &\quad \text{for all } (x,y) \in X_3^-, \end{aligned}$$

that is

$$\begin{aligned} w_1 + w_2 \qquad\quad + w_4 &> 0, && (15) \\ w_1 - w_2 \qquad\quad + w_4 &> 0, && (16) \\ w_1 \qquad\quad + w_3 + w_4 &< 0, && (17) \\ w_1 \qquad\quad - w_3 + w_4 &< 0. && (18) \end{aligned}$$

Adding (15) and (16), we obtain $w_1 + w_4 > 0$, while adding (17) and (18) yields $w_1 + w_4 < 0$, which establishes the contradiction.